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Singular Integrals and Related Topics

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SINGULAR INTEGRALS AND RELATED TOPICS

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Preface

It is well-known that singular integrals is continuously regarded as a central role in harmonic analysis. There are many nice books related to singular integrals. In this book, there are at least two sides which differ from the other books. One of them is to establish more perfect theory of singular integrals. It includes not only the case of smooth kernels, but also the case of rough kernels. In the same way, we deal with some related operators, such as fractional integral operators and Littlewood-Paley operators. The other is to introduce more new theory on some oscillatory singular integrals with polynomial phases. This book is mainly provided to graduate students in analysis field. However, it is also beneficial to researchers in mathematics.

This book consists of five chapters. Let us now illustrate the choice of material in each chapter. Chapter 1 is devoted to the theory of the Hardy-Littlewood maximal operator as the basis of singular integrals and other related operators. It also includes the basic theory of the A_p weights. Chapter 2 is related to the theory of singular integrals. Since the theory of singular integrals with Calderón-Zygmund kernel has been introduced in many books, we will pay more attention to the singular integrals with homogeneous kernels. Specially, we will introduce more perfect theory of singular integrals with rough kernels, for instance the L^p boundedness of singular integrals with kernels in certain Hardy space on the unit sphere will be fully proved. In addition, the weighted L^p boundedness of singular integrals with rough kernels and their commutators will be also established. Chapter 3 is devoted to fractional integrals. In the same way, we will pay more attention to the case of rough kernels. It includes not only the $A(p, q)$ weight theory of fractional integrals with rough kernels, but also the theory of its commutators. Chapter 4 is to introduce a class of oscillatory singular integrals with polynomial phases. Note that this oscillatory singular integral is neither a Calderón-Zygmund operator nor a convolution operator. However there exists certain link between this oscillatory singular integral and the corresponding singular integral. Therefore, the criterion on the L^p boundedness of oscillatory singular integrals will become a crucial role in this chapter. It will discover an equivalent relation between the L^p boundedness of the oscillatory singular integral and that of the corresponding truncated singular integral. Chapter 5 is related to the Littlewood-Paley theory. In this chapter, we will establish two kinds of the weakest conditions on the kernel for the L^p boundedness of Marcinkiewicz integral operator with rough kernel. Finally, it is worth pointing out that as space is limited, the theory of singular integrals and related operators in this book is only worked on the Lebesgue spaces although there are many good results on other spaces such as Hardy spaces and BMO space.

It should be pointed out that many results in the later three chapters of this book reflect the research accomplishment by the authors of this book and their cooperators. We would like to acknowledge to Jiecheng Chen, Dashan Fan, Yongsheng Han, Yingsheng Jiang, Chin-Cheng Lin, Guozhen Lu, Yibiao Pan, Fernando Soria and Kozo Yabuta for their effective cooperates in the study of singular integrals. On this occasion, the authors deeply cherish the memory of Minde Cheng and Yongsheng Sun for their constant encourage. The first named author of this book, Shanzhen Lu, would like to express his thanks to his former students Wengu Chen, Yong Ding, Zunwei Fu, Yiqing Gui, Guoen Hu, Junfeng Li, Guoquan Li, Xiaochun Li, Yan Lin, Heping Liu, Mingju Liu, Zhixin Liu, Zongguang Liu, Bolin Ma, Huixia Mo, Lin Tang, Shuangping Tao, Huoxiong Wu, Qiang Wu, Xia Xia, Jingshi Xu, Qingying Xue, Dunyan Yan, Dachun Yang, Pu Zhang, and Yan Zhang for their cooperations and contributions to the study of harmonic analysis during the joint working period. Finally, Shanzhen Lu would like to express his deep gratitude to Guido Weiss for his constant encourage and help.

Shanzhen Lu
Yong Ding
Dunyan Yan

December, 2006

Contents

Chapter 1

Hardy-Littlewood Maximal Operator

1.1 Hardy-Littlewood maximal operator

Let us begin with giving the definition of the Hardy-Littlewood maximal function, which plays a very important role in harmonic analysis.

Definition 1.1.1 (Hardy-Littlewood maximal function) *Suppose that f is a locally integrable on $\mathbb{R}^n$, i.e., $f \in L^1_{loc}(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$, the Hardy-Littlewood maximal function $Mf(x)$ of f is defined by*

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\leq r} |f(x-y)|dy. \tag{1.1.1}$$

Moreover, M is also called as the Hardy-Littlewood maximal operator.

Sometimes we need to use the following maximal functions. For $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M'f(x) = \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|dy, \tag{1.1.2}$$

where and below, $Q(x,r)$ denotes the cube with the center at x and with side r and its sides parallel to the coordinate axes. Moreover, $|E|$ denotes the Lebesgue measure of the set E. More general,

$$M''f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy, \tag{1.1.3}$$

where the supremum is taken over all cubes or balls Q containing x.

For the Hardy-Littlewood maximal operator M, we would like to give the following some remarks.

Remark 1.1.1 *By (1.1.1)-(1.1.3), it is easy to see that there exist constants C_i $(i = 0, 1, 2, 3)$ depending only on the dimension n such that*

$$C_0 Mf(x) \le C_1 M'f(x) \le C_2 M''f(x) \le C_3 Mf(x) \tag{1.1.4}$$

for any $x \in \mathbb{R}^n$. That is, the Hardy-Littlewood maximal function Mf of f and the maximal functions $M'f$, $M''f$ are pointwise equivalent each other.

Remark 1.1.2 *For $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function $Mf(x)$ is a lower semi-continuous function on $\mathbb{R}^n$, and is then a measurable function on $\mathbb{R}^n$.*

By (1.1.4), we only need to show it for $M'f(x)$. In fact, it is sufficient to show that for any $\lambda \in \mathbb{R}$, the set $E = \{x \in \mathbb{R}^n : M'f(x) > \lambda\}$ is an open set. However, by the definition of $M'f(x)$ it suffices to show that E is open for all $\lambda > 0$. Equivalently, we only need to show that $E^c := \{x \in \mathbb{R}^n : M'f(x) \le \lambda\}$ is a closed set for all $\lambda > 0$.

Suppose that $\{x_k\} \subset E^c$ satisfying $x_k \to x$ as $k \to \infty$. We only need to show that for any $r > 0$

$$\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|dy \le \lambda. \tag{1.1.5}$$

Denote $Q_k = Q(x_k, r)$ and $f_k(y) = f(y)\chi_{Q(x,r) \triangle Q_k}(y)$ for all $k = 1, 2, \cdots$, where

$$Q(x,r) \triangle Q_k = \big(Q(x,r) \backslash Q_k\big) \bigcup \big(Q_k \backslash Q(x,r)\big).$$

Thus,

$$|f_k(y)| \le |f(y)| \quad \textit{for all} \quad k \quad \textit{and} \quad \lim_{k \to \infty} f_k(y) = 0.$$

Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{k \to \infty} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f_k(y)|dy = 0. \tag{1.1.6}$$

On the other hand, it is clear that

$$\frac{1}{|Q(x,r)|}\int_{Q_k}|f(y)|dy = \frac{1}{|Q_k|}\int_{Q_k}|f(y)|dy \le \lambda.$$

Hence

$$\begin{aligned}\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f(y)|dy &\le \frac{1}{|Q(x,r)|}\int_{Q(x,r)\triangle Q_k}|f(y)|dy \\ &\quad+\frac{1}{|Q(x,r)|}\int_{Q_k}|f(y)|dy \\ &\le \frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f_k(y)|dy+\lambda.\end{aligned}$$

Let $k\to\infty$, by (1.1.6) we obtain (1.1.5).

Remark 1.1.3 *The Hardy-Littlewood maximal operator M is not a bounded operator from $L^1(\mathbb{R}^n)$ to itself.*

We only consider the case $n = 1$. Take $f(x) = \chi_{[0,1]}(x)$, then for any $x \ge 1$, we have

$$Mf(x) \ge \frac{1}{2x}\int_0^{2x}|f(y)|dy = \frac{1}{2x}.$$

Hence

$$\int_{\mathbb{R}} Mf(x)dx \ge \int_1^\infty Mf(x)dx \ge \int_1^\infty \frac{1}{2x}dx = \infty.$$

Although M is not a bounded operator on $L^1(\mathbb{R}^n)$, however, as its a replacement result we shall see that M is a bounded operator from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, i.e., the weak $L^1(\mathbb{R}^n)$ space (see Definition 1.1.2 below).

Lemma 1.1.1 (Vitali type covering lemma) *Let E be a measurable subset of $\mathbb{R}^n$ and let $\mathcal{B}$ be a collection of balls B with bounded diameter $d(B)$ covering E in Vitali's sense, i.e. for any $x \in E$ there exist a ball $B_x \in \mathcal{B}$ such that $x \in B_x$. Then there exist a $\beta > 0$ depending only on n, and disjoint countable balls $B_1, B_2, \cdots, B_k, \cdots$ in $\mathcal{B}$ such that*

$$\sum_k |B_k| \ge \beta|E|.$$

In fact, it will be seen from the proof below that it suffices to take $\beta = 5^{-n}$.

Proof. Denote $\ell_0 = \sup\{d(B) : B \in \mathcal{B}\} < \infty$. Take $B_1 \in \mathcal{B}$ so that $d(B_1) \geq \frac{1}{2}\ell_0$. Again denote $\mathcal{B}_1 = \{B : B \in \mathcal{B}$ and $B \cap B_1 = \emptyset\}$ and $\ell_1 = \sup\{d(B) : B \in \mathcal{B}_1\}$, then we choose $B_2 \in \mathcal{B}_1$ such that $d(B_2) \geq \frac{1}{2}\ell_1$.

Suppose that $B_1, B_2, \cdots, B_k$ have been chosen from $\mathcal{B}$ according to the above way, then we denote

$$\mathcal{B}_k = \left\{ B : B \in \mathcal{B} \;\text{ with }\; B \bigcap \left(\bigcup_{j=1}^{k} B_j \right) = \emptyset \right\}$$

and

$$\ell_k = \sup\{d(B) : B \in \mathcal{B}_k\}.$$

Next we choose $B_{k+1} \in \mathcal{B}_k$ such that $d(B_{k+1}) \geq \frac{1}{2}\ell_k$. Thus we may choose a sequence $B_1, B_2, \cdots,$ from $\mathcal{B}$ such that

(i) $B_1, B_2, \cdots, B_k, \cdots$ are disjoint;

(ii) $d(B_{k+1}) \geq \frac{1}{2}\sup\{d(B) : B \in \mathcal{B}_k\}$, and

$$\mathcal{B}_k = \left\{ B : B \in \mathcal{B} \;\text{ and }\; B \bigcap \left(\bigcup_{j=1}^{k} B_j \right) = \emptyset \right\}$$

for $k = 1, 2, \cdots$.

If this process stops at some B_k, then it shows that $\mathcal{B}_k = \emptyset$. In this case, for any $x \in E$ there exists a ball $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \cap B_{k_0} \neq \emptyset$ with some $1 \leq k_0 \leq k$. Without loss of generality, we may assume that $B_x \cap B_j = \emptyset$ for $j = 1, 2, \cdots, k_0 - 1$. So, $d(B_{k_0}) \geq \frac{1}{2}d(B_x)$, and this implies $B_x \subset 5B_{k_0}$, where $5B_{k_0}$ expresses the five times extension of B_{k_0} with the same center. Thus, we have $E \subset \bigcup_{j=1}^{k} 5B_j$, and it leads to

$$|E| \leq \left| \bigcup_{j=1}^{k} 5B_j \right| \leq \sum_{j=1}^{k} |5B_j| \leq 5^n \sum_{j=1}^{k} |B_j|.$$

On the other hand, it is trivial when $\sum_{j=1}^{\infty} |B_k| = \infty$. So, we may assume that $\sum_{j=1}^{\infty} |B_k| < \infty$. Denote $B_k^* = 5B_k$. We will claim that

$$E \subset \bigcup_{k=1}^{\infty} B_k^*. \tag{1.1.7}$$

In fact, it suffices to prove that $B \subset \bigcup_{k=1}^{\infty} B_k^*$ for any $B \in \mathcal{B}$. Since $\sum_{j=1}^{\infty} |B_k| < \infty$, we have $d(B_k) \to 0$ as $k \to \infty$. Thus there exists k_0 such that $d(B_{k_0}) < \frac{1}{2} d(B)$. Of course, we may think that the index k_0 is the smallest with the above property. In this case, B_x must intersect with some B_j for $1 \le j \le k_0 - 1$. Otherwise, $d(B_{k_0}) \ge \frac{1}{2} d(B_x)$. As before, we get $B_x \subset 5B_j = B_j^*$ and (1.1.7) follows. Thus

$$|E| \le \left| \bigcup_{k=1}^{\infty} B_k^* \right| \le \sum_{k=1}^{\infty} |B_k^*| \le 5^n \sum_{k=1}^{\infty} |B_k|.$$

This completes the proof. ■

Definition 1.1.2 (Weak L^p spaces) *Suppose that $1 \le p < \infty$ and f is a measurable function on $\mathbb{R}^n$. The function f is said to belong to the weak L^p spaces on $\mathbb{R}^n$, if there is a constant $C > 0$ such that*

$$\sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{1/p} \le C < \infty.$$

In other words, the weak $L^p(\mathbb{R}^n)$ is defined by

$$L^{p,\infty}(\mathbb{R}^n) = \big\{ f : \ \|f\|_{p,\infty} < \infty \big\},$$

where

$$\|f\|_{p,\infty} := \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{1/p}$$

denotes the seminorm of f in the weak $L^p(\mathbb{R}^n)$.

Remark 1.1.4 *It is easy to verify that for $1 \le p < \infty$, $L^p(\mathbb{R}^n) \subsetneqq L^{p,\infty}(\mathbb{R}^n)$.*

Definition 1.1.3 (Operator of type $(p,\ q)$) *Suppose that T is a sublinear operator and $1 \le p,\ q \le \infty$. T is said to be of weak type (p, q) if T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$. That is, there exists a constant $C > 0$ such that for any $\lambda > 0$ and $f \in L^p(\mathbb{R}^n)$*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \le \left(\frac{C}{\lambda} \|f\|_p \right)^q; \tag{1.1.8}$$

T is said to be of type (p, q) if T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. That is, there exists a constant $C > 0$ such that for any $f \in L^p(\mathbb{R}^n)$

$$\|Tf\|_q \le C \|f\|_p, \tag{1.1.9}$$

where and below, $\|f\|_p = \|f\|_{L^p(\mathbb{R}^n)}$ denotes the L^p norm of $f(x)$.

When $p = q$ and the operator T satisfies (1.1.8) or (1.1.9), T is also said to be of weak type (p, p), respectively. Moreover, It is easy to see that an operator of type (p, q) is also of weak type (p, q), but its reverse is not hold generally.

Below we shall prove that the Hardy-Littlewood maximal operator M is of weak type $(1, 1)$ and type (p, p) for $1 < p \leq \infty$, respectively.

Theorem 1.1.1 *Let f be a measurable function on $\mathbb{R}^n$.*

(a) If $f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, then $Mf(x) < \infty$ a.e. $x \in \mathbb{R}^n$.

(b) There exists a constant $C = C(n) > 0$ such that for any $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

(c) There exists a constant $C = C(n, p) > 0$ such that for any $f \in L^p(\mathbb{R}^n)$ $1 < p \leq \infty$, $\|Mf\|_p \leq C\|f\|_p$.

Proof. Obviously, the conclusion (a) is a direct result of the conclusions (b) and (c). Hence we only give the proof of (b) and (c).

Let us first consider (b). For any $\lambda > 0$, by Remark 1.1.1 the set

$$E_\lambda := \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$$

is an open set, and is then a measurable set. By Definition 1.1.1, for any $x \in E_\lambda$, there exists a ball B_x with the center at x such that

$$\frac{1}{|B_x|}\int_{B_x} |f(y)|dy > \lambda.$$

Thus

$$|B_x| < \frac{1}{\lambda}\int_{B_x} |f(y)|dy \leq \frac{1}{\lambda}\|f\|_1 < \infty \quad \text{for all} \quad x \in E_\lambda.$$

Therefore, if we denote $\mathcal{B} = \{B_x : x \in E_\lambda\}$, then $\mathcal{B}$ covers E_λ in Vitali's sense. By Lemma 1.1.1, we may choose disjoint countable balls $B_1, B_2, \cdots$, $B_k, \cdots$ in $\mathcal{B}$ such that

$$\sum_k |B_k| \geq \beta|E_\lambda|.$$

Hence

$$\begin{aligned}
\beta|E_\lambda| \leq \sum_k |B_k| &\leq \frac{1}{\lambda}\sum_k \int_{B_k} |f(y)|dy \\
&= \frac{1}{\lambda}\int_{\bigcup_k B_k} |f(y)|dy \\
&\leq \frac{1}{\lambda}\|f\|_1.
\end{aligned}$$

Let us now turn to the proof of (c). Clearly, the conclusion (c) holds for $p=\infty$, we only consider the case $1<p<\infty$. Let $f\in L^p(\mathbb{R}^n)$ $(1<p<\infty)$. For any $\lambda>0$, write $f=f_1+f_2$, where

$$f_1(x)=\begin{cases} f(x), & \text{for} \quad |f(x)|\geq \lambda/2 \\ 0, & \text{for} \quad |f(x)|<\lambda/2. \end{cases}$$

It is easy to see that $f_1\in L^1(\mathbb{R}^n)$. Thus we have

$$|f(x)|\leq |f_1(x)|+\frac{\lambda}{2} \quad \text{and} \quad Mf(x)\leq Mf_1(x)+\frac{\lambda}{2}. \tag{1.1.10}$$

Hence, by (1.1.10) and the weak (1,1) boundedness of M (i.e. the conclusion (b)), we have

$$\begin{aligned} |E_\lambda| &= \big|\{x\in\mathbb{R}^n : Mf(x)>\lambda\}\big| \\ &\leq \big|\{x\in\mathbb{R}^n : Mf_1(x)>\lambda/2\}\big| \\ &\leq \frac{2\beta}{\lambda}\int_{\mathbb{R}^n}|f_1(x)|\,dx \\ &= \frac{2\beta}{\lambda}\int_{\{x\in\mathbb{R}^n:\ |f(x)|\geq\lambda/2\}}|f(x)|\,dx, \end{aligned}$$

where β is the constant in Lemma 1.1.1. Therefore

$$\begin{aligned} &\int_{\mathbb{R}^n}(Mf(x))^p dx \\ &= p\int_0^\infty \lambda^{p-1}|E_\lambda|\,d\lambda \\ &\leq p\int_0^\infty \lambda^{p-1}\left(\frac{2\beta}{\lambda}\int_{\{x\in\mathbb{R}^n:\ |f(x)|\geq\lambda/2\}}|f(x)|\,dx\right)d\lambda \\ &\leq 2\beta p\int_{\mathbb{R}^n}|f(x)|\left(\int_0^{2|f(x)|}\lambda^{p-2}d\lambda\right)dx \\ &= \frac{2\beta p}{p-1}\int_{\mathbb{R}^n}|f(x)|^p\,dx. \end{aligned}$$

Thus we finish the proof of Theorem 1.1.1. ■

Immediately, by the weak (1,1) boundedness of the Hardy-Littlewood maximal operator M we may get the Lebesgue differentiation theorem.

Theorem 1.1.2 (Lebesgue differentiation theorem) *Suppose that* $f\in L^1_{loc}(\mathbb{R}^n)$. *Then*

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}f(y)\,dy=f(x) \quad a.e. \ \ x\in\mathbb{R}^n,$$

where $B(x,r)$ denotes the ball with the center at x and radius r.

Proof. Since for any $R>0$, $f\chi_{B(0,R)} \in L^1(\mathbb{R}^n)$, we may assume that $f \in L^1(\mathbb{R}^n)$. Denote

$$L_r(f)(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy,$$

and let

$$\Lambda(f)(x) = \overline{\lim_{r\to 0}} L_r(f)(x) - \underline{\lim}_{r\to 0} L_r(f)(x).$$

Then

$$\Lambda(f)(x) \le 2\sup_{r>0}|L_r(f)(x)| = 2Mf(x).$$

Let us first show that for any $\lambda > 0$

$$|E_\lambda(\Lambda f)| := |\{x \in \mathbb{R}^n : \Lambda(f)(x) > \lambda\}| = 0. \tag{1.1.11}$$

In fact, for any $\varepsilon > 0$ we may decompose $f = g + h$, where g is a continuous function with compact support set and $\|h\|_1 < \varepsilon$. Thus

$$\Lambda(f)(x) \le \Lambda(g)(x) + \Lambda(h)(x) = \Lambda(h)(x),$$

and it leads to

$$|E_\lambda(\Lambda f)| \le |E_\lambda(\Lambda h)| \le |E_{\lambda/2}(Mh)|.$$

By Theorem 1.1.1 (b), we have

$$|E_\lambda(\Lambda f)| \le \frac{2C}{\lambda}\|h\|_1 < \frac{2C\varepsilon}{\lambda}.$$

Thus, by the arbitrariness of ε we know (1.1.11) holds, and (1.1.11) shows that the limit $\lim\limits_{r\to 0} L_r(f)(x)$ exists for a.e. $x \in \mathbb{R}^n$.

On the other hand, by the integral continuity, we have

$$\begin{aligned}
&\lim_{r\to 0}\|L_r(f) - f\|_1 \\
&= \lim_{r\to 0}\int_{\mathbb{R}^n}\left|\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy - f(x)\right|dx \\
&= \lim_{r\to 0}\int_{\mathbb{R}^n}\frac{1}{|B(0,r)|}\left|\int_{B(0,r)}[f(x-y) - f(x)]dy\right|dx \\
&\le \lim_{r\to 0}\frac{1}{|B(0,r)|}\int_{B(0,r)}\int_{\mathbb{R}^n}|f(x-y) - f(x)|dx\,dy = 0.
\end{aligned}$$

Hence there exists a subsequence $\{r_k\}$ satisfying $r_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to\infty} L_{r_k}(f)(x) = f(x) \quad \text{for a.e. } \ x \in \mathbb{R}^n.$$

Because $\lim\limits_{r\to 0} L_r(f)(x)$ exists for a.e. $x \in \mathbb{R}^n$, thus

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy = f(x) \quad \text{a.e. } \ x \in \mathbb{R}^n,$$

which is the desired conclusion. ■

Remark 1.1.5 *Clearly, by the equivalence of (1.1.1) - (1.1.3), it is easy to see that the conclusion of Theorem 1.1.2 still holds if we replace the ball $B(x,r)$ by cube $Q(x,r)$, even more generally, by a cube Q containing x.*

1.2 Calderón-Zygmund decomposition

Applying Lebesgue differentiation theorem, we may give a decomposition of $\mathbb{R}^n$, called as Calderón-Zygmund decomposition, which is extremely useful in harmonic analysis.

Theorem 1.2.1 (Calderón-Zygmund decomposition of $\mathbb{R}^n$) *Suppose that f is nonnegative integrable on $\mathbb{R}^n$. Then for any fixed $\lambda > 0$, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes (here by disjoint we mean that their interiors are disjoint) such that*

(1) $f(x) \le \lambda$ *for a.e.* $x \notin \bigcup_j Q_j$;

(2) $\left|\bigcup_j Q_j\right| \le \frac{1}{\lambda}\|f\|_1$;

(3) $\lambda < \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \le 2^n\lambda$.

Proof. By $f \in L^1(\mathbb{R}^n)$, we may decompose $\mathbb{R}^n$ into a net of equal cubes with whose interiors are disjoint such that for every Q in the net

$$\frac{1}{|Q|}\int_Q f(x)\,dx \le \lambda.$$

Let Q' be any fixed cube in the net. We divide it into 2^n equal cubes, and denote Q'' is one of these cubes. Then there are the follwing two cases.
Case (i)

$$\frac{1}{|Q''|}\int_{Q''} f(x)\,dx > \lambda.$$

Case (ii)

$$\frac{1}{|Q''|}\int_{Q''} f(x)\,dx \le \lambda.$$

In the case (i) we have

$$\lambda < \frac{1}{|Q''|}\int_{Q''} f(x)\,dx \le \frac{1}{2^{-n}|Q'|}\int_{Q'} f(x)\,dx \le 2^n\lambda.$$

Hence, we do not sub-divide Q'' any further, and Q'' is chosen as one of the sequence $\{Q_j\}$.

For the case (ii) we continuously sub-divide Q'' into 2^n equal subcubes, and repeat this process until we are forced into the case (i). Thus we get a sequence $\{Q_j\}$ of cubes obtained from the case (i). By Theorem 1.1.2,

$$f(x) = \lim_{\substack{Q\ni x\\ |Q|\to 0}} \frac{1}{|Q|}\int_Q f(x)\,dx \le \lambda \quad \text{for a.e.} \quad x \notin \bigcup_j Q_j.$$

This proves the theorem. ■

Remark 1.2.1 *In place of $\mathbb{R}^n$ by a fixed cube Q_0, we may similarly discuss the Calderón-Zygmund decomposition on Q_0 for $f \in L^1(Q_0)$ and $\lambda > 0$. Moreover, we also may obtain the similar decomposition for $f \in L^p(\mathbb{R}^n)$ $(p > 1)$.*

An application of the Calderón-Zygmund decomposition on $\mathbb{R}^n$ is that it may be used to give the L^1 boundedness of the Hardy-Littlewood maximal operator M in some sense. More precisely, we have the following conclusion.

Theorem 1.2.2 *Suppose that $f \in L^1(\mathbb{R}^n)$.*

(i) If

$$\int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|dx < \infty,$$

then $Mf \in L^1_{loc}(\mathbb{R}^n)$.

(ii) If f is supported in a ball B and $Mf \in L^1(B)$, then

$$\int_B |f(x)| \log^+ |f(x)| dx < \infty,$$

where

$$\log^+ |f(x)| = \begin{cases} \log |f(x)|, & \text{for } |f(x)| > 1 \\ 0, & \text{for } |f(x)| \leq 1. \end{cases}$$

Proof. We first give two estimates on the maximal operator.

$$|\{x \in \mathbb{R}^n : Mf(x) > 2\lambda\}| \leq \frac{C}{\lambda} \int_{\{x:\ |f(x)|>\lambda\}} |f(x)| dx, \tag{1.2.1}$$

and

$$|\{x \in \mathbb{R}^n : M''f(x) > \lambda\}| \geq \frac{C}{\lambda} \int_{\{x:\ |f(x)|>\lambda\}} |f(x)| dx. \tag{1.2.2}$$

The constant C appearing in (1.2.1) and (1.2.2) depends only on the dimension n. In fact, (1.2.1) has appeared in the proof of Theorem 1.1.1. As for (1.2.2), using the Calderón-Zygmund decomposition on $\mathbb{R}^n$ (Theorem 1.2.1) for $\lambda > 0$, we get the sequence $\{Q_j\}$ of cubes to satisfy

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|\, dx \leq 2^n \lambda,$$

and $f(x) \leq \lambda$ for a.e. $x \notin \bigcup_j Q_j$. Thus for $x \in Q_j$, $M''f(x) > \lambda$ and

$$\begin{aligned} |\{x \in \mathbb{R}^n : M''f(x) > \lambda\}| &\geq \sum_j |Q_j| \\ &\geq \frac{1}{2^n \lambda} \sum_j \int_{Q_j} |f(x)|\, dx \\ &\geq \frac{1}{2^n \lambda} \int_{\{x:|f(x)|>\lambda\}} |f(x)|\, dx. \end{aligned}$$

Let us return to the proof of Theorem 1.2.2. First we consider (i). For any compact set $K \subset \mathbb{R}^n$, it follows from (1.2.1) that

$$\begin{aligned}
\int_K Mf(x)dx &= \int_0^\infty |\{x \in K : Mf(x) > \lambda\}|d\lambda \\
&= \int_0^\infty 2|\{x \in K : Mf(x) > 2\lambda\}|d\lambda \\
&\le 2\left\{ \int_0^1 |K|d\lambda + \int_1^\infty |\{x \in \mathbb{R}^n : Mf(x) > 2\lambda\}|d\lambda \right\} \\
&\le 2\left\{ |K| + \int_1^\infty \frac{C}{\lambda} \int_{\{x:|f(x)|>\lambda\}} |f(x)|dxd\lambda \right\} \\
&= 2\left\{ |K| + C\int_{\mathbb{R}^n} |f(x)| \int_1^{|f(x)|} \frac{d\lambda}{\lambda} dx \right\} \\
&= 2\left\{ |K| + C\int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)|dx \right\} \\
&< \infty.
\end{aligned}$$

Thus we obtain the conclusion (i).

As for (ii), without loss of generality, we may assume that the radius of B is R. we denote by B' the ball with center as one of B and the radius $2R$. For any $x' \in B' \setminus B$, we take $x \in B$ such that x is the point symmetric to x' with respect to the boundary of B. Then it is easy to check that for any $r > 0$, $B(x', r) \cap B \subset B(x, 10r)$. Thus,

$$\begin{aligned}
\frac{1}{|B(x', r)|} \int_{B(x',r)} |f(y)|dy &= \frac{1}{|B(x', r)|} \int_{B(x',r)\cap B} |f(y)|dy \\
&\le C_n \frac{1}{|B(x, 10r)|} \int_{B(x,10r)} |f(y)|dy \\
&\le C_n Mf(x).
\end{aligned}$$

Hence $Mf(x') \le C_n Mf(x)$ for any $x' \in B' \setminus B$. By the estimates above, it is easy to see that $Mf \in L^1(B')$. On the other hand, for any $x \in \mathbb{R}^n \setminus B'$, when $r < R$ we have

$$\frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)|dy = 0$$

by the support set condition. Thus Mf is bounded on $\mathbb{R}^n \setminus B'$, since

$$Mf(x) \le \frac{1}{|B(x, R)|} \|f\|_1, \quad \text{for any} \quad x \in \mathbb{R}^n \setminus B'.$$

It is also clear that

$$Mf(x) \to 0, \quad \text{as} \quad |x| \to \infty.$$

Thus, for any $\lambda_0 > 0$ the set $\{x \in \mathbb{R}^n : Mf(x) > \lambda_0\}$ must be contained in some ball. So

$$I = \int_{\{x\in\mathbb{R}^n : Mf(x)>\lambda_0\}} Mf(x)dx < \infty.$$

Now, if we take $\lambda_0 = C_n$, then we have

$$I \geq \int_{C_n}^{\infty} |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| d\lambda.$$

On the other hand, by (1.1.4) $Mf(x) \geq C_n M''f(x)$ for any $x \in \mathbb{R}^n$. If we denote $\lambda' = \lambda/C_n$, then by (1.2.2)

$$\begin{aligned}
|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| &\geq |\{x \in \mathbb{R}^n : M''f(x) > \lambda'\}| \\
&\geq \frac{C}{\lambda'} \int_{\{x\colon\ |f(x)|>\lambda'\}} |f(x)|dx \\
&= \frac{C_n C}{\lambda} \int_{\{x\colon\ C_n|f(x)|>\lambda\}} |f(x)|dx.
\end{aligned}$$

By two inequalities above we have

$$\begin{aligned}
I &\geq \int_{C_n}^{\infty} \frac{c_n C}{\lambda} \int_{\{x\colon\ c_n|f(x)|>\lambda\}} |f(x)|dx\, d\lambda \\
&= c_n C \int_{\mathbb{R}^n} |f(x)| \int_{c_n}^{c_n|f(x)|} \frac{d\lambda}{\lambda} dx \\
&= c_n C \int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)| dx.
\end{aligned}$$

The proof of Theorem 1.2.2 is finished. ∎

Another important application of the Calderón-Zygmund decomposition on $\mathbb{R}^n$ is that it may be used to deduce the Calderón-Zygmund decomposition on function. The later is a very important tool in harmonic analysis.

Theorem 1.2.3 (Calderón-Zygmund decomposition for function) *Let f be nonnegative integrable on $\mathbb{R}^n$. Then for any fixed $\lambda > 0$, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes and the functions g, b such that* *(i)* $f(x) = g(x) + b(x)$;

(ii) $|g(x)| \le 2^n\lambda$ *for a.e.* $x \in \mathbb{R}^n$;

(iii) $\|g\|_p^p \le C\lambda^{p-1}\|f\|_1$ *for* $1 < p < \infty$;

(iv) $b(x) = 0$, *a.e.* $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$;

(v) $\displaystyle\int_{Q_j} b(x)\,dx = 0, \quad j = 1, 2, \cdots.$

Proof. Applying the Calderón-Zygmund decomposition on $\mathbb{R}^n$ for f and $\lambda > 0$, we obtain a sequence $\{Q_j\}$ of disjoint dyadic cubes such that

$$f(x) \le \lambda \quad \text{for a.e.} \quad x \in \mathbb{R}^n \setminus \bigcup_j Q_j;$$

$$\sum_j |Q_j \le \frac{1}{\lambda}\|f\|_1;$$

and

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \le 2^n\lambda, \quad \text{for} \quad j = 1, 2, \cdots.$$

Now we define $g(x)$ and $b(x)$ as follows.

$$g(x) = \begin{cases} f(x) & x \in \mathbb{R}^n \setminus \bigcup_j Q_j \\ \frac{1}{|Q_j|}\displaystyle\int_{Q_j} f(x)\,dx, & x \in Q_j, j = 1, 2, \cdots; \end{cases}$$

and

$$b(x) = \begin{cases} 0 & x \in \mathbb{R}^n \setminus \bigcup_j Q_j \\ f(x) - \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f(x)\,dx, & x \in Q_j, j = 1, 2, \cdots. \end{cases}$$

Thus $f(x) = g(x) + b(x)$, and by the definition of b, both (iv) and (v) hold. Moreover, by the definition of g, (ii) holds. Finally, if denote $F = \mathbb{R}^n \setminus \bigcup_j Q_j$, then

$$\begin{aligned}\|g\|_p^p &= \sum_j \int_{Q_j} g(x)^{p-1} g(x)\,dx + \int_F g(x)^{p-1} g(x)\,dx \\ &\le \sum_j (2^n\lambda)^{p-1} \int_{Q_j} g(x)\,dx + \lambda^{p-1} \int_F g(x)\,dx \\ &= (2^n\lambda)^{p-1} \sum_j \int_{Q_j} f(x)\,dx + \lambda^{p-1} \int_F f(x)\,dx \\ &\le C\lambda^{p-1}\|f\|_1.\end{aligned}$$

This is just (iii), and the proof of Theorem 1.2.3 is finished. ■

1.3 Marcinkiewicz interpolation theorem

Notice that in the proof of L^p boundedness of the Hardy-Littlewood maximal operator M, we only use the weak L^1 and L^∞ boundedness of M. This inspires us to consider the similar problem for more general sublinear operator.

Theorem 1.3.1 (Marcinkiewicz interpolation theorem) *Let $(X, \mathcal{X}, \mu)$ and $(X, \mathcal{Y}, \nu)$ be two measure spaces, and let the sublinear operator T be both of weak type (p_0, p_0) and weak type (p_1, p_1) for $1 \le p_0 < p_1 \le \infty$. That is, there exist the constants $C_0, C_1 > 0$ such that for any $\lambda > 0$ and f*

(i) $\nu(\{x \in X : |Tf(x)| > \lambda\}) \le \left(\dfrac{C_0}{\lambda}\|f\|_{p_0,\mu}\right)^{p_0}$;

(ii) $\nu(\{x \in X : |Tf(x)| > \lambda\}) \le \left(\dfrac{C_1}{\lambda}\|f\|_{p_1,\mu}\right)^{p_1}$ *for $p_1 < \infty$.*

If $p_1 = \infty$, then the weak type and strong type coincide by definition:

$$\|Tf\|_{\infty,\nu} \le C_1\|f\|_{\infty,\mu}.$$

Then T is also of type (p, p) for all $p_0 < p < p_1$, i.e. there exist a constants $C > 0$ such that for any $f \in L^p(X, \mu)$

$$\left(\int_X |Tf(x)|^p d\nu\right)^{1/p} \le C\left(\int_X |f(x)|^p d\mu\right)^{1/p}.$$

Proof. For $f \in L^p(X,\mu)$ and $\lambda > 0$, write $f(x) = f^\lambda(x) + f_\lambda(x)$, where

$$f^\lambda(x) = \begin{cases} f(x), \text{ for } & |f(x)| > \lambda, \\ 0, \text{ for } & |f(x)| \le \lambda. \end{cases}$$

Thus, $f^\lambda \in L^{p_0}(X,\mu)$ and $f_\lambda \in L^{p_1}(X,\mu)$. Moreover,

$$|Tf(x)| \le |Tf^\lambda(x)| + |Tf_\lambda(x)|.$$

Case I: $p_1 < \infty$. By the weak type (p_i, p_i) of T $(i = 1, 2)$, we have

$$\begin{aligned} \nu(\{x \in X : |Tf(x)| > \lambda\}) &\le \nu\Big(\{x \in X : |Tf^\lambda(x)| > \lambda/2\}\Big) \\ &\quad + \nu\Big(\{x \in X : |Tf_\lambda(x)| > \lambda/2\}\Big) \\ &\le \left(\frac{2C_0}{\lambda}\|f^\lambda\|_{p_0,\mu}\right)^{p_0} + \left(\frac{2C_1}{\lambda}\|f_\lambda\|_{p_1,\mu}\right)^{p_1}. \end{aligned} \tag{1.3.1}$$

By (1.3.1), we have that

$$\begin{aligned} \int_X |Tf(x)|^p d\nu &= p\int_0^\infty \lambda^{p-1}\nu(\{x \in X : |Tf(x)| > \lambda\})d\lambda \\ &\le 2^{p_0}C_0 p\int_0^\infty \lambda^{p-p_0-1}\int_{\{x\in X:|f(x)|>\lambda\}} |f(x)|^{p_0}d\mu(x)d\lambda \\ &\quad + 2^{p_1}C_1 p\int_0^\infty \lambda^{p-p_1-1}\int_{\{x\in X:|f(x)|<\lambda\}} |f(x)|^{p_1}d\mu(x)d\lambda \\ &\le 2^{p_0}C_0 p\int_X |f(x)|^{p_0}\int_0^{|f(x)|} \lambda^{p-p_0-1}d\lambda d\mu(x) \\ &\quad + 2^{p_1}C_1 p\int_X |f(x)|^{p_1}\int_{|f(x)|}^\infty \lambda^{p-p_1-1}d\lambda d\mu(x) \\ &= \frac{2^{p_0}C_0 p}{p-p_0}\int_X |f(x)|^p d\mu + \frac{2^{p_1}C_1 p}{p_1-p}\int_X |f(x)|^p d\mu \\ &= C_p\int_X |f(x)|^p d\mu. \end{aligned}$$

Case II: $p_1 = \infty$. In this case, if we denote $\alpha = 1/(2C_1)$, then

$$\nu(\{x \in X : |Tf(x)| > \lambda\}) \le \nu(\{x \in X : |Tf^{\alpha\lambda}(x)| > \lambda/2\})$$

and

$$\begin{aligned}\int_X |Tf(x)|^p d\nu &\le 2^{p_0} C_0 p \int_0^\infty \lambda^{p-p_0-1} \int_{\{x\in X: |f(x)|>\alpha\lambda\}} |f(x)|^{p_0} d\mu(x) d\lambda \\ &= 2^{p_0} C_0 p \int_X |f(x)|^{p_0} \int_0^{|f(x)|/\alpha} \lambda^{p-p_0-1} d\lambda d\mu(x) \\ &= C_p \int_X |f(x)|^p d\mu.\end{aligned}$$

This completes the proof of Theorem 1.3.1. ∎

As an application of the Marcinkiewicz interpolation theorem, below we will prove the Fefferman-Stein inequality on the Hardy-Littlewood maximal operator M.

Theorem 1.3.2 (Fefferman-Stein inequality) *Let* $1 < p < \infty$. *Then there exists a constant* $C = C_{n,p}$ *such that for any nonnegative measurable function* $\varphi(x)$ *on* $\mathbb{R}^n$ *and* f*,*

$$\int_{\mathbb{R}^n} (Mf(x))^p \varphi(x) dx \le C \int_{\mathbb{R}^n} |f(x)|^p M\varphi(x) dx. \tag{1.3.2}$$

Proof. Without loss of generality, we may assume that $M\varphi(x) < \infty$ a.e. $x \in \mathbb{R}^n$ and $M\varphi(x) > 0$. If denote

$$d\mu(x) = M\varphi(x)dx \quad \text{and} \quad d\nu(x) = \varphi(x)dx,$$

then by the Marcinkiewicz interpolation theorem in order to get (1.3.2), it suffices to prove that M is both of type $(L^\infty(\mu), L^\infty(\nu))$ and of weak type $(L^1(\mu), L^1(\nu))$.

Let us first show that M is of type $(L^\infty(\mu), L^\infty(\nu))$. In fact, if $\|f\|_{\infty,\mu} \le a$, then

$$\int_{\{x\in\mathbb{R}^n:\ |f(x)|>a\}} M\varphi(x)dx = \mu(\{x \in \mathbb{R}^n :\ |f(x)| > a\}) = 0.$$

Since $M\varphi(x) > 0$ for any $x \in \mathbb{R}^n$, we have $|\{x \in \mathbb{R}^n :\ |f(x)| > a\}| = 0$, equivalently, $|f(x)| \le a$ a.e. on $\mathbb{R}^n$. Thus $Mf(x) \le a$ a.e. on $\mathbb{R}^n$ and this follows $\|Mf\|_{\infty,\nu} \le a$. Therefore, $\|Mf\|_{\infty,\nu} \le \|f\|_{\infty,\mu}$.

Before proving that M is also of weak type $(L^1(\mu), L^1(\nu))$, we give the following lemma.

Lemma 1.3.1 *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. If the sequence $\{Q_k\}$ of cubes is chosen from the Calderón-Zygmund decomposition on $\mathbb{R}^n$ for f and $\lambda > 0$, then*

$$\{x \in \mathbb{R}^n : \ M'f(x) > 7^n\lambda\} \subset \bigcup_k Q_k^*,$$

where $Q_k^* = 2Q_k$. Then we have

$$|\{x \in \mathbb{R}^n : \ M'f(x) > 7^n\lambda\}| \le 2^n \sum_k |Q_k|.$$

Proof. Suppose that $x \notin \bigcup_k Q_k^*$. Then there are two cases for any cube Q with the center x. If $Q \subset \mathbb{R}^n \setminus \bigcup_k Q_k$, then

$$\frac{1}{|Q|}\int_Q |f(x)|dx \le \lambda.$$

If $Q \cap Q_k \ne \emptyset$ for some k, then it is easy to check that $Q_k \subset 3Q$, and

$$\bigcup_k \{Q_k : \ Q_k \cap Q \ne \emptyset\} \subset 3Q.$$

Hence, for $F = \mathbb{R}^n \setminus \bigcup_k Q_k$, we have

$$\begin{aligned}
\int_Q |f(x)|dx &\le \int_{Q\cap F} |f(x)|dx + \sum_{Q_k \cap Q \ne \emptyset} \int_{Q_k} |f(x)|dx \\
&\le \lambda|Q| + \sum_{Q_k \cap Q \ne \emptyset} 2^n\lambda|Q_k| \\
&\le \lambda|Q| + 2^n\lambda|3Q| \\
&\le 7^n\lambda|Q|.
\end{aligned}$$

Thus we know that $M'f(x) \le 7^n\lambda$ for any $x \notin \bigcup_k Q_k^*$, and it yields that

$$|\{x \in \mathbb{R}^n : \ M'f(x) > 7^n\lambda\}| \le \left|\bigcup_k Q_k^*\right| = 2^n \sum_k |Q_k|.$$

∎

Let us return to the proof of weak type $(L^1(\mu), L^1(\nu))$. We need to prove that there exists a constant C such that for any $\lambda > 0$ and $f \in L^1(\mu)$

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^n: \ Mf(x)>\lambda\}} \varphi(x)dx &= \nu(\{x \in \mathbb{R}^n : \ Mf(x) > \lambda\}) \\
&\le \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|M\varphi(x)dx.
\end{aligned} \tag{1.3.3}$$

We may assume that $f \in L^1(\mathbb{R}^n)$. In fact, if take $f_k = |f|\chi_{B(0,k)}$, then $f_k \in L^1(\mathbb{R}^n)$, $0 \leq f_k(x) \leq f_{k+1}(x)$ for $x \in \mathbb{R}^n$ and $k = 1, 2, \cdots$. Moreover, $\lim_{k\to\infty} f_k(x) = |f(x)|$.

By (1.1.4), there exists $c_n > 0$ such that $Mf(x) \leq c_n M'f(x)$ for all $x \in \mathbb{R}^n$. Applying the Calderón-Zygmund decomposition on $\mathbb{R}^n$ for f and $\lambda' = \frac{\lambda}{c_n 7^n}$, we get a sequence $\{Q_k\}$ of cubes satisfying

$$\lambda' < \frac{1}{|Q_k|}\int_{Q_k} |f(x)|dx \leq 2^n\lambda'.$$

By Lemma 1.3.1, we have that

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^n:\ Mf(x)>\lambda\}} \varphi(x)dx &\leq \int_{\{x\in\mathbb{R}^n:\ M'f(x)>7^n\lambda'\}} \varphi(x)dx \\
&\leq \int_{\bigcup_k Q_k^*} \varphi(x)dx \leq \sum_k \int_{Q_k^*} \varphi(x)dx \\
&\leq \sum_k \left(\frac{1}{|Q_k|}\int_{Q_k^*}\varphi(x)dx\right)\left(\frac{1}{\lambda'}\int_{Q_k}|f(y)|dy\right) \\
&= \frac{c_n 7^n}{\lambda}\sum_k \int_{Q_k} |f(y)|\left(\frac{2^n}{|Q_k^*|}\int_{Q_k^*}\varphi(x)dx\right)dy \\
&\leq \frac{c_n 14^n}{\lambda}\sum_k \int_{Q_k} |f(y)|M''\varphi(y)dy \\
&= \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(y)|M\varphi(y)dy.
\end{aligned}$$

Thus, M is of weak type $(L^1(\mu), L^1(\nu))$, and the Fefferman-Stein inequality can be obtained by applying Theorem 1.3.1 with $p_0 = 1$ and $p_1 = \infty$. ∎

The Marcinkiewicz interpolation theorem shows that if a sublinear operator T is the weak boundedness of at two "ends" of indexes, then T is also the strong boundedness for all index point between the two ends. The following conclusion shows that the weak boundedness of a sublinear operator T at an index point is equivalent to its strong boundedness in some sense for all index which are less than this point.

Theorem 1.3.3 (Kolmogorov inequality) *Suppose that T is a sublinear operator from $L^p(\mathbb{R}^n)$ to measurable function spaces and $1 \leq p < \infty$. Then*

(a) *If T is of weak type (p,p), then for all $0 < r < p$ and all set E with finite measure, there exists a constant $C > 0$ such that*

$$\int_E |Tf(x)|^r dx \leq C|E|^{1-r/p}\|f\|_p^r. \tag{1.3.4}$$

(b) *If there exists $0 < r < p$ and constant $C > 0$, such that (1.3.4) holds for all set E with finite measure and $f \in L^p(\mathbb{R}^n)$, then T is of weak type (p,p).*

Proof of (a). Since T is of weak type (p,p), for any $\lambda > 0$

$$|\{x \in \mathbb{R}^n : \ |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda^p}\|f\|_p^p.$$

Thus for any set E with $|E| < \infty$

$$|\{x \in E : \ |Tf(x)| > \lambda\}| \leq |\{x \in \mathbb{R}^n : \ |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda^p}\|f\|_p^p.$$

Hence for any $0 < r < p$, we have

$$\begin{aligned}
&\int_E |Tf(x)|^r dx \\
&= r\int_0^\infty \lambda^{r-1}|\{x \in E : \ |Tf(x)| > \lambda\}|d\lambda \\
&\leq r\int_0^\infty \lambda^{r-1}\min\{|E|, \ \frac{C}{\lambda^p}\|f\|_p^p\}d\lambda \\
&\leq r\int_0^{C\|f\|_p|E|^{-1/p}} \lambda^{r-1}|E|d\lambda + Cr\int_{C\|f\|_p|E|^{-1/p}}^\infty \lambda^{r-1-p}\|f\|_p^p d\lambda \\
&\leq C|E|^{1-r/p}\|f\|_p^r.
\end{aligned}$$

Proof of (b). For any $\lambda > 0$, take $E = \{x \in \mathbb{R}^n : \ |Tf(x)| > \lambda\}$, then E is a measurable set and $|E| < \infty$. Otherwise, there is a sequence $\{E_k\}$ of measurable sets such that $E_k \subset E$ and $|E_k| = k$ for $k = 1, 2, \cdots$. Thus for every k, we have

$$\lambda^r k = \lambda^r |E_k| \leq \int_{E_k} |Tf(x)|^r dx \leq C|E_k|^{1-r/p}\|f\|_p^r = Ck^{1-r/p}\|f\|_p^r.$$

However, it is not true. Thus by (1.3.4), we obtain

$$\lambda^r|E| \leq \int_{\{x\in\mathbb{R}^n: \ |Tf(x)|>\lambda\}} |Tf(x)|^r dx \leq C|E|^{1-r/p}\|f\|_p^r.$$

From this it follows

$$|\{x \in \mathbb{R}^n : \ |Tf(x)| > \lambda\}| = |E| \leq \frac{C}{\lambda^p}\|f\|_p^p.$$

Hence T is of weak type (p,p). ■

1.4 Weighted norm inequalities

In this section we shall extend the conclusions of Theorem 1.1.1 to a general measure spaces $(\mathbb{R}^n, \omega(x)dx)$, where ω is called a weight function, which is a nonnegative locally integrable function on $\mathbb{R}^n$.

For the sake of convenience, in this section we shall use (1.1.3) as the definition of the Hardy-Littlewood maximal operator M. Indeed, by (1.1.4) the maximal operator M'' is essentially the same as the operator M. Now we give the definition of A_p weight.

Definition 1.4.1 (A_p weights $(1 \leq p < \infty)$) *Let $\omega(x) \geq 0$ and $\omega(x) \in L^1_{loc}(\mathbb{R}^n)$. We say that $\omega \in A_p$ for $1 < p < \infty$ if there is a constant $C > 0$ such that*

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} \leq C, \tag{1.4.1}$$

where and below, $1/p + 1/p' = 1$. We say that $\omega \in A_1$ if there is a constant $C > 0$ such that

$$M\omega(x) \leq C\omega(x). \tag{1.4.2}$$

The smallest constant appearing in (1.4.1) or (1.4.2) is called the A_p constant of ω.

Remark 1.4.1 *Clearly, $\omega \in A_1$ if and only if there is a constant $C > 0$ such that for any cube Q*

$$\frac{1}{|Q|}\int_Q \omega(x)dx \leq C \inf_{x\in Q} \omega(x), \tag{1.4.3}$$

where and below, inf *is the essential infimum. Moreover, it is easy to see that for $1 \leq p < \infty$ and any $\omega \in A_p$, its A_p constant $C \geq 1$. In fact, for any* *cube Q, we have that*

$$
\begin{aligned}
1 &= \frac{1}{|Q|}\int_Q \omega(x)^{1/p}\cdot\omega(x)^{-1/p}dx \\
&\le \left\{\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1}\right\}^{1/p} \\
&\le C^{1/p}.
\end{aligned}
$$

Let us now give some elementary properties of A_p weights.

Proposition 1.4.1 (Properties (I) of A_p weights)

(i) $A_p \subsetneqq A_q$, *if* $1 \le p < q < \infty$.

(ii) For $1 < p < \infty$, $\omega \in A_p$ *if and only if* $\omega(x)^{1-p'} \in A_{p'}$.

(iii) If $\omega_0,\ \omega_1 \in A_1$, *then* $\omega_0\omega_1^{1-p} \in A_p$ *for* $1 < p < \infty$.

(iv) If $\omega \in A_p$ $(1 \le p < \infty)$, *then for any* $0 < \varepsilon < 1$, $\omega^\varepsilon \in A_p$.

(v) If $\omega \in A_p$ $(1 \le p < \infty)$, *then for any* $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\left(\frac{1}{|Q|}\int_Q |f(x)|dx\right)^p \cdot \omega(Q) \le C\int_Q |f(x)|^p\omega(x)dx. \tag{1.4.4}$$

(vi) If $\omega \in A_p$ $(1 \le p < \infty)$, *then for any* $\delta > 1$ *there exists a constant* $C(n,p,\delta)$ *such that for any cube* Q, $\omega(\delta Q) \le C(n,p,\delta)\omega(Q)$. *In particular, the case taking* $\delta = 2$ *shows that* A_p *weights satisfy double condition.*

(vii) If $\omega \in A_p$ $(1 \le p < \infty)$, *then for any* $0 < \alpha < 1$ *there exists* $0 < \beta < 1$ *such that for any measurable subset* $E \subset Q$, $|E| \le \alpha|Q|$ *and* $\omega(E) \le \beta\omega(Q)$.

Proof. (i) For $p > 1$, this is a direct consequence of Definition 1.4.1 and Hölder's inequality. If $p = 1$, then by (1.4.3)

$$
\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-q'}dx\right)^{q-1} &\le \sup_{x\in Q}\omega(x)^{-1} \\
&= \big(\inf_{x\in Q}\omega(x)\big)^{-1} \\
&\le C\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)^{-1}.
\end{aligned}
$$

On the other hand, since $|x|^\alpha \in A_p$ if and only if $-n < \alpha < n(p-1)$ (see Proposition 1.4.4 in this chapter), we have $A_p \neq A_q$.

(ii) Since $(p-1)(p'-1) = 1$, for any cube Q we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)\left(\frac{1}{|Q|}\int_Q [\omega(x)^{1-p'}]^{1-p}\,dx\right)^{p'-1} \\
&\qquad = \left\{\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1}\right\}^{p'-1} \\
&\qquad \leq C^{p'-1}.
\end{aligned}$$

(iii) For any Q, by $\omega_i \in A_1$ $(i = 1, 2)$ and (1.4.3) we have

$$\omega_i(x)^{-1} \leq \sup_{x\in Q} \omega_i(x)^{-1} \leq \big(\inf_{x\in Q} \omega_i(x)\big)^{-1} \leq C\left(\frac{\omega_i(Q)}{|Q|}\right)^{-1}. \tag{1.4.5}$$

From (1.4.5), it immediately follows that

$$\left(\frac{1}{|Q|}\int_Q \omega_0(x)\omega_1(x)^{1-p}\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega_0(x)^{1-p'}\omega_1(x)\,dx\right)^{p-1} \leq C.$$

(iv) For $p = 1$, by Hölder's inequality and (1.4.3)

$$\frac{1}{|Q|}\int_Q \omega(x)^\varepsilon dx \leq \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)^\varepsilon \leq \big(C\inf_{x\in Q}\omega(x)\big)^\varepsilon = C^\varepsilon \inf_{x\in Q}\omega(x)^\varepsilon.$$

Similarly, we can get the conclusion (iv) for $1 < p < \infty$ by Definition 1.4.1 and Hölder's inequality.

(v) For $p = 1$, by (1.4.3), we have that

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(x)|dx \cdot \omega(Q) &= \int_Q |f(x)|dx \cdot \frac{1}{|Q|}\int_Q \omega(x)dx \\
&\leq C\int_Q |f(x)|dx \cdot \inf_{x\in Q}\omega(x) \\
&\leq C\int_Q |f(x)|\omega(x)dx.
\end{aligned}$$

When $1 < p < \infty$, by Hölder's inequality

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |f(x)|dx \\
&= \frac{1}{|Q|}\int_Q |f(x)|\omega(x)^{1/p}\omega(x)^{-1/p}dx \\
&\le \left(\frac{1}{|Q|}\int_Q |f(x)|^p\omega(x)dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{(p-1)/p} \\
&\le C^{1/p}\left(\frac{1}{|Q|}\int_Q |f(x)|^p\omega(x)dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)^{-1/p}.
\end{aligned}$$

Thus, (1.4.4) follows from this.

(vi) If we replace Q by δQ and $f(x)$ by $\chi_Q(x)$ in (1.4.4), respectively, then this is just the conclusion of (vi).

(vii) Let $S = Q \setminus E$ and $f(x) = \chi_S(x)$ in (1.4.3). Then

$$\left(\frac{|S|}{|Q|}\right)^p \omega(Q) \le C\int_S \omega(x)dx.$$

Thus

$$(1-\alpha)^p\omega(Q) \le \left(1-\frac{|E|}{|Q|}\right)^p\omega(Q) \le C\left(\int_Q \omega(x)dx - \int_E \omega(x)dx\right).$$

Notice that $C \ge 1$ (see Remark 1.4.1),

$$\omega(E) \le \frac{C-(1-\alpha)^p}{C}\omega(Q).$$

So, we get the conclusion of (vii) with $\beta = (C-(1-\alpha)^p)/C$. ■

The following theorem gives a very important and useful property of A_p weights.

Theorem 1.4.1 (Reverse Hölder inequality) *Let $\omega \in A_p$, $1 \le p < \infty$. Then there exist a constant C and $\varepsilon > 0$ depending only on p and the A_p constant of ω, such that for any cube Q*

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} \le \frac{C}{|Q|}\int_Q \omega(x)dx. \tag{1.4.6}$$

Proof. Fix a cube Q, by Remark 1.2.1 we apply the Calderón-Zygmund decomposition with respect to Q for ω and the increasing sequence

$$\{\omega(Q)/|Q| = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots\}$$

For each λ_k we may get a sequence $\{Q_{k,i}\}$ of disjoint cubes such that

$$\omega(x) \le \lambda_k \quad \text{for} \quad x \notin \Lambda_k = \bigcup_i Q_{k,i},$$

and

$$\lambda_k < \frac{1}{|Q_{k,i}|}\int_{Q_{k,i}} \omega(x)dx \le 2^n \lambda_k.$$

Since $\lambda_{k+1} > \lambda_k$, for every $Q_{k+1,j}$ it is either to equal some $Q_{k,i}$ or a subcube of $Q_{k,i}$ for some i. Thus

$$\begin{aligned}
|Q_{k+1,j}| &< \frac{1}{\lambda_{k+1}}\int_{Q_{k+1,j}} \omega(x)dx \\
&= \frac{|Q_{k,i}|}{\lambda_{k+1}}\frac{1}{|Q_{k,i}|}\int_{Q_{k+1,j}} \omega(x)dx \\
&\le \frac{|Q_{k,i}|}{\lambda_{k+1}}\frac{1}{|Q_{k,i}|}\int_{Q_{k,i}} \omega(x)dx \\
&\le 2^n \frac{\lambda_k}{\lambda_{k+1}}|Q_{k,i}|.
\end{aligned}$$

From this, it follows

$$|Q_{k,i} \cap \Lambda_{k+1}| \le 2^n \frac{\lambda_k}{\lambda_{k+1}}|Q_{k,i}|.$$

For fixed $\alpha < 1$, we choose $\{\lambda_k\}$ such that $2^n\lambda_k/\lambda_{k+1} = \alpha$. Equivalently, $\lambda_k = (2^n/\alpha)^k\lambda_0$. Thus

$$|Q_{k,i} \cap \Lambda_{k+1}| \le \alpha|Q_{k,i}|.$$

By Proposition 1.4.1 (vii), there exists $0 < \beta < 1$ such that

$$\omega(Q_{k,i} \cap \Lambda_{k+1}) \le \beta\omega(Q_{k,i}).$$

Summing up with respect to the index i, we obtain $\omega(\Lambda_{k+1}) \le \beta\omega(\Lambda_k)$, and it yields

$$\omega(\Lambda_{k+1}) \le \beta^k\omega(\Lambda_0).$$

Similarly, we also have $|\Lambda_{k+1}| \le \alpha|\Lambda_k|$ and $|\Lambda_{k+1}| \le \alpha^k|\Lambda_0|$. Hence

$$\left|\bigcap_{k=0}^{\infty} \Lambda_k\right| = \lim_{k\to\infty} |\Lambda_k| = 0.$$

Thus

$$\begin{aligned}
\int_Q \omega(x)^{1+\varepsilon}dx &= \int_{Q\setminus\Lambda_0} \omega(x)^{1+\varepsilon}dx + \sum_{k=0}^{\infty}\int_{\Lambda_k\setminus\Lambda_{k+1}} \omega(x)^{1+\varepsilon}dx \\
&\le \lambda_0^{\varepsilon}\omega(Q\setminus\Lambda_0) + \sum_{k=0}^{\infty} \lambda_{k+1}^{\varepsilon}\omega(\Lambda_k\setminus\Lambda_{k+1}) \\
&\le \lambda_0^{\varepsilon}\Big(\omega(Q\setminus\Lambda_0) + \sum_{k=0}^{\infty}(2^n/\alpha)^{(k+1)\varepsilon}\beta^k\omega(\Lambda_0)\Big) \\
&\le \lambda_0^{\varepsilon}\Big(\omega(Q\setminus\Lambda_0) + (2^n/\alpha)^{\varepsilon}\sum_{k=0}^{\infty}[(2^n/\alpha)^{\varepsilon}\beta]^k\omega(\Lambda_0)\Big).
\end{aligned}$$

Let $\varepsilon > 0$ be small enough such that $(2^n/\alpha)^\varepsilon\beta < 1$. Then the series converges. Therefore we have

$$\begin{aligned}
\int_Q \omega(x)^{1+\varepsilon}dx &\le C\lambda_0^{\varepsilon}\big(\omega(Q\setminus\Lambda_0) + \omega(\Lambda_0)\big)) \\
&= C\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)^{\varepsilon} \cdot \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)|Q|.
\end{aligned}$$

Thus we get (1.4.6) and the proof of Theorem 1.4.1 is finished. ∎

As a corollary of Theorem 1.4.1, we get some further properties of A_p weights.

Proposition 1.4.2 (Properties (II) of A_p weights)

(viii) *If* $\omega \in A_p$ $(1 < p < \infty)$, *then there is an* $\varepsilon > 0$ *such that* $p - \varepsilon > 1$ *and* $\omega(x) \in A_{p-\varepsilon}$.

(ix) $A_p = \bigcup_{q<p} A_q$, *if* $1 < p < \infty$.

(x) *If* $\omega \in A_p$ $(1 \le p < \infty)$, *then there is an* $\varepsilon > 0$ *such that* $\omega(x)^{1+\varepsilon} \in A_p$.

(xi) *If* $\omega \in A_p$ $(1 \le p < \infty)$,*then there is* $\delta > 0$ *and* $C > 0$ *such that for any cube* Q *and a measurable subset* $E \subset Q$

$$\frac{\omega(E)}{\omega(Q)} \le C\left(\frac{|E|}{|Q|}\right)^{\delta}. \tag{1.4.7}$$

Proof. (viii) Let $\omega \in A_p$, then $\omega^{1-p'} \in A_{p'}$ by the property (ii). Applying the Reverse Hölder inequality (Theorem 1.4.1) for $\omega^{1-p'}$, then

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^{(1-p')(1+\theta)}dx\right)^{(p-1)/(1+\theta)} \le C^{p-1}\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1},$$

where $\theta > 0$. Now multiplying the factor

$$\frac{1}{|Q|}\int_Q \omega(x)dx$$

on two sides of the above inequality, we have

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{(1-p')(1+\theta)}dx\right)^{(p-1)/(1+\theta)} \le C.$$

Denote $(1-p')(1+\theta) = 1-q'$, then $1 < q < p$ and $\omega \in A_q$. Thus we get the property (viii) with $\varepsilon = p - q$.

(ix) The property (ix) is a direct corollary of the properties (i) and (viii). Indeed, we know that $A_p \supset \bigcup_{q<p} A_q$ by the property (i). On the other hand, by (viii), $A_p \subset \bigcup_{q<p} A_q$.

(x) Suppose that $\omega \in A_1$, then by (1.4.6)

$$\frac{1}{|Q|}\int_Q \omega(x)^{1+\varepsilon}dx \le \left(\frac{C}{|Q|}\int_Q \omega(x)dx\right)^{1+\varepsilon} \le C\cdot\omega(x)^{1+\varepsilon} \quad \text{a.e. } x \in \mathbb{R}^n.$$

Hence $\omega(x)^{1+\varepsilon} \in A_1$.

If $\omega \in A_p,\ p > 1$, then $\omega(x)^{1-p'} \in A_{p'}$ by (ii). Using Theorem 1.3.3, it is clear that there exists an $\varepsilon > 0$ such that

$$\frac{1}{|Q|}\int_Q \omega(x)^{1+\varepsilon}dx \le C_1\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)^{1+\varepsilon}$$

and

$$\frac{1}{|Q|}\int_Q \omega(x)^{(1-p')(1+\varepsilon)}dx \le C_2\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{1+\varepsilon}.$$

hold at the same time. Thus,

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \omega(x)^{1+\varepsilon}dx\right)\left(\frac{1}{|Q|}\int_Q [\omega(x)^{1+\varepsilon}]^{1-p'}dx\right)^{p-1}\\ &\le C\left\{\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1}\right\}^{1+\varepsilon} \le C.\end{aligned}$$

This shows that $\omega^{1+\varepsilon} \in A_p$.

(xi) Since $\omega(x) \in A_p$ $(1 \le p < \infty)$, using Hölder's inequality for $1+\varepsilon$ and $(1+\varepsilon)/\varepsilon$, where ε is fixed by (1.4.6), we have

$$\begin{aligned}\int_E \omega(x)dx &\le \left(\int_E \omega(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} \cdot |E|^{\varepsilon/(1+\varepsilon)} \\ &= \left(\frac{1}{|Q|}\int_E \omega(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} |Q|^{1/(1+\varepsilon)}|E|^{\varepsilon/(1+\varepsilon)} \\ &\le \frac{C}{|Q|}\int_Q \omega(x)dx|Q|^{1/(1+\varepsilon)}|E|^{\varepsilon/(1+\varepsilon)} \\ &= C\omega(Q)\left(\frac{|E|}{|Q|}\right)^{\varepsilon/(1+\varepsilon)}.\end{aligned}$$

Thus we have showed that ω satisfies (1.4.7) if taking $\delta = \varepsilon/(1+\varepsilon)$. ■

Remark 1.4.2 *Let ω be a nonnegative locally integrable function on $\mathbb{R}^n$. We say that $\omega \in A_\infty$ if ω satisfies (1.4.7). The property (xi) shows that $\bigcup_{1\le p<\infty} A_p \subset A_\infty$. However, it can be proved that the above containing relationship may be reversed. So, we have indeed $A_\infty = \bigcup_{1\le p<\infty} A_p$.*

In this section, we shall see that A_p weights give a characterization of weighted weak L^p and strong L^p boundedness for the Hardy-Littlewood maximal operator M.

Theorem 1.4.2 (Characterization of the weighted weak type (p,p)) *Suppose that $1 \le p < \infty$. Then the Hardy-Littlewood maximal operator M is of weak type $(L^p(\omega dx), L^p(\omega dx))$ if and only if $\omega \in A_p$.*

That is, there exists a constant $C > 0$ such that for any $\lambda > 0$ and $f(x) \in L^p(\omega dx)$ $(1 \le p < \infty)$

$$\int_{\{x\in\mathbb{R}^n:\ Mf(x)>\lambda\}} \omega(x)dx \le \frac{C}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx \tag{1.4.8}$$

if and only if $\omega \in A_p$.

Proof. Let us first prove that $\omega \in A_p$ is the necessary condition of (1.4.8). For $p = 1$, let Q be any cube and $Q_1 \subset Q$. If denote $f = \chi_{Q_1}$, then for any $0 < \lambda < |Q_1|/|Q|$, we have $Q \subset \{x \in \mathbb{R}^n :\ Mf(x) > \lambda\}$. By (1.4.8)

$$\lambda\int_Q \omega(x)dx \le \lambda\int_{\{x\in\mathbb{R}^n:\ Mf(x)>\lambda\}} \omega(x)dx \le C\int_{Q_1} \omega(x)dx.$$

By the arbitrariness of $\lambda < |Q_1|/|Q|$, we get

$$\frac{1}{|Q|}\int_Q \omega(x)dx \le \frac{C}{|Q_1|}\int_{Q_1} \omega(x)dx.$$

Applying Lebesgue differentiation theorem (Theorem 1.1.2), we have that

$$\frac{1}{|Q|}\int_Q \omega(y)dy \le C\omega(x) \quad \text{ a.e. } \ x \in Q.$$

By the arbitrariness of cube Q again, we have

$$M\omega(x) \le C\omega(x) \quad \text{ a.e. } \ x \in \mathbb{R}^n.$$

Thus $\omega \in A_1$.

For $p > 1$, let Q be any cube. If we take $f = \omega^{1-p'}\chi_Q$, then for any $0 < \lambda < \omega^{1-p'}(Q)/|Q|$, $Q \subset \{x \in \mathbb{R}^n : \ Mf(x) > \lambda\}$. By (1.4.8)

$$\begin{aligned}\frac{\lambda^p}{|Q|}\int_Q \omega(x)dx &\le \frac{\lambda^p}{|Q|}\int_{\{x\in\mathbb{R}^n:\ Mf(x)>\lambda\}} \omega(x)dx \\ &\le \frac{C}{|Q|}\int_{\mathbb{R}^n} [\omega(x)^{1-p'}\chi_Q(x)]^p\omega(x)dx \\ &= \frac{C}{|Q|}\int_Q \omega(x)^{1-p'}dx.\end{aligned}$$

Hence by arbitrariness of λ, we get

$$\frac{1}{|Q|}\int_Q \omega(x)dx \le C\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{1-p}.$$

So $\omega \in A_p$.

Below we will prove that $\omega \in A_p$ is also a sufficient condition of (1.4.8). When $p = 1$, (1.4.8) is just a direct corollary of (1.3.3). Now let us consider the case $p > 1$. By Hölder's inequality and the condition $\omega \in A_p$, we have

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q |f(x)|dx\right)^p \\ &= \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q |f(x)|\,\omega(x)^{1/p}\omega(x)^{-1/p}dx\right)^p \\ &\le \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q |f(x)|^p\omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1} \\ &\le C\frac{1}{|Q|}\int_Q |f(x)|^p\omega(x)dx.\end{aligned}$$

Thus, this shows that if $f \in L^p(\omega dx)$ then $f \in L^1_{loc}(\mathbb{R}^n)$, and

$$\int_Q \omega(x)dx \le C\left(\frac{1}{|Q|}\int_Q |f(x)|dx\right)^{-p}\left(\int_Q |f(x)|^p\omega(x)dx\right). \tag{1.4.9}$$

So without loss of generality, we may assume that $f \ge 0$ and $f \in L^1(\mathbb{R}^n)$. Applying the Calderón-Zygmund decomposition (Theorem 1.2.1) for f at height $7^{-n}\lambda$, we get a cube sequence $\{Q_k\}$ such that

$$7^{-n}\lambda < \frac{1}{|Q_k|}\int_{Q_k} f(x)dx \quad \text{for all} \quad Q_k. \tag{1.4.10}$$

By Lemma 1.3.1, we have that

$$\{x \in \mathbb{R}^n : \ Mf(x) > \lambda\} \subset \bigcup_k Q_k^*,$$

where $Q_k^* = 2Q_k$. Thus by the property (vi) of A_p weights and (1.4.9) and (1.4.10), we conclude that

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^n:\ Mf(x)>\lambda\}} \omega(x)dx \\
&\le \sum_k \int_{2Q_k} \omega(x)dx \le C2^{np}\sum_k \int_{Q_k} \omega(x)dx \\
&\le C2^{np}\sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} |f(x)|dx\right)^{-p}\left(\int_{Q_k} |f(x)|^p\omega(x)dx\right) \\
&\le C\frac{14^{np}}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx.
\end{aligned}$$

This shows that the Hardy-Littlewood maximal operator M is of weak type $(L^p(\omega dx), L^p(\omega dx))$. Hence we finish the proof of Theorem 1.4.2. ■

Theorem 1.4.3 (Characterization of the weighted strong type (p,p)) *Suppose that $1 < p < \infty$. Then the Hardy-Littlewood maximal operator M is of strong type $(L^p(\omega dx), L^p(\omega dx))$ if and only if $\omega \in A_p$.*

Proof. The necessity is a corollary of Theorem 1.4.2. In fact, since M is of strong type $(L^p(\omega dx), L^p(\omega dx))$, and is also weak type $(L^p(\omega dx), L^p(\omega dx))$. Then by Theorem 1.4.2, $\omega \in A_p$.

On the other hand, if $\omega \in A_p$ for $1 < p < \infty$, then by the property (viii) there exists $1 < q < p$ such that $\omega \in A_q$. By Theorem 1.4.2, M is of weak type $(L^p(\omega dx), L^p(\omega dx))$. That is,

$$\int_{\{x\in\mathbb{R}^n:\ Mf(x)>\lambda\}} \omega(x)dx \le \frac{C}{\lambda^q}\int_{\mathbb{R}^n} |f(x)|^q\omega(x)dx. \tag{1.4.11}$$

Notice that if $\omega \in A_p$, then for any measurable set $E \in \mathbb{R}^n$ $\omega(E) = 0$ if and only if $|E| = 0$. So, $L^\infty(\omega dx) = L^\infty$ in the sense of equality of norms. Hence by the $L^\infty(\mathbb{R}^n)$-boundedness of M (Theorem 1.1.1), we have

$$\|Mf\|_{\infty,\omega} \le \|f\|_{\infty,\omega}.$$

Using the Marcinkiewicz interpolation theorem (Theorem 1.3.1) between this and (1.4.11) we obtain

$$\int_{\mathbb{R}^n} Mf(x)^p\omega(x)dx \le C\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx.$$

In other words, M is of strong type $(L^p(\omega dx), L^p(\omega dx))$. ∎

In this section, we shall discuss further some important properties of A_1 weights, such as the construction of A_1 weights, the relationship between A_1 weights and A_p weights and its some application, etc.

Theorem 1.4.4 (Constructive characterization of A_1 weights)

(a) If $f(x) \in L^1_{loc}(\mathbb{R}^n)$ and $Mf(x) < \infty$, a.e. $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal operator. Then for any $0 < \varepsilon < 1$, $\omega(x) = \Big(Mf(x)\Big)^\varepsilon \in A_1$ and whose A_1 constant depends only on ε.

(b) If $\omega(x) \in A_1$, then there exists $f(x) \in L^1_{loc}(\mathbb{R}^n)$, $0 < \varepsilon < 1$ and a function $b(x)$ such that

(i) $0 < C_1 \le b(x) \le C_2$, a.e. $x \in \mathbb{R}^n$;

(ii) $\Big(Mf(x)\Big)^\varepsilon < \infty$, a.e. $x \in \mathbb{R}^n$;

(iii) $\omega(x) = b(x)\Big(Mf(x)\Big)^\varepsilon$.

Proof of (a). By Remark 1.4.1, we need to show that for any $0 < \varepsilon < 1$, there exists a constant C such that for any cube Q in $\mathbb{R}^n$

$$\frac{1}{|Q|}\int_Q \Big(Mf(y)\Big)^\varepsilon dy \le C\Big(Mf(x)\Big)^\varepsilon \quad \text{for a.e. } \ x \in Q.$$

Fix Q and $0 \le \varepsilon < 1$, we denote $f = f\chi_{2Q} + f\chi_{\mathbb{R}^n\setminus 2Q} := f_1 + f_2$. Then

$$\Big(Mf(x)\Big)^\varepsilon \le \Big(Mf_1(x)\Big)^\varepsilon + \Big(Mf_2(x)\Big)^\varepsilon.$$

By the weak $(1,1)$ boundedness of M and Kolmogorov's inequality (Theorem 1.3.3)

$$\begin{aligned}\frac{1}{|Q|}\int_Q Mf_1(x)^\varepsilon dx &\le \frac{C}{|Q|}|Q|^{1-\varepsilon}\|f_1\|_1^\varepsilon \\ &\le C\bigg(\frac{1}{|Q|}\int_{2Q}|f_1(y)|dy\bigg)^\varepsilon \\ &\le C2^{n\varepsilon}\Big(Mf(x)\Big)^\varepsilon\end{aligned}$$

for any $x \in Q$.

Now we give the estimate of $\Big(Mf_2(x)\Big)^\varepsilon$. Clearly, if $y \in Q$ and a cube $Q' \ni y$ such that $Q' \cap (\mathbb{R}^n \setminus 2Q) \ne \emptyset$, then $4Q' \supset Q$. Hence if $x \in Q$

$$\frac{1}{|Q'|}\int_{Q'}|f_2(y)|dy \le \frac{4^n}{|4Q'|}\int_{4Q'}|f_2(y)|dy \le 4^n Mf(x).$$

This shows that $Mf_2(y) \le 4^n Mf(x)$ for any $y \in Q$, and so is

$$\frac{1}{|Q|}\int_Q \Big(Mf_2(y)\Big)^\varepsilon dy \le 4^{n\varepsilon}\Big(Mf(x)\Big)^\varepsilon.$$

Proof of (b). Since $\omega(x) \in A_1$, then by the Reverse Hölder inequality (Theorem 1.4.1) there exists $\eta > 0$ such that for any cube Q

$$\bigg(\frac{1}{|Q|}\int_Q \omega(x)^{1+\eta}dx\bigg)^{1/(1+\eta)} \le \frac{1}{|Q|}\int_Q \omega(x)dx \le C\omega(x).$$

This implies that

$$M(\omega^{1+\eta})(x)^{1/(1+\eta)} \le C\omega(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^n.$$

On the other hand, by the Lebesgue differentiation theorem (Theorem 1.1.2)

$$\omega(x)^{1+\eta} \le M(\omega^{1+\eta})(x).$$

Therefore, if let $\varepsilon = 1/(1+\eta)$, $f(x) = \omega(x)^{1+\eta}$ and $b(x) = \omega(x)/\Big(Mf(x)\Big)^{\varepsilon}$, then we obtain the conclusions of (b). Thus we finish the proof of Theorem 1.4.4. ■

Below we shall turn to discuss the relationship between A_1 weights and A_p weights.

By Remark 1.4.1, if $\omega \in A_1$, then we have the following equivalent form of (1.4.2)

$$\frac{1}{|Q|}\int_Q \omega(x)dx \cdot \sup_{x\in Q}(\omega(x)^{-1}) \le C,$$

where and below, sup is the essential supremun. On the other hand, it is easy to see that when $p \to 1$

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1} \to \|\omega^{-1}\|_{\infty,Q} = \sup_{x\in Q}(\omega(x)^{-1}).$$

Therefore, the A_1 weights can be seen as the limit of A_p weights when $p \to 1$.

By the property (iii) of A_p weights, we have known that if $\omega_1, \omega_2 \in A_1$ then $\omega_1 \cdot \omega_2^{1-p} \in A_p$. A very deep result is that the converse of above conclusion is also true.

Theorem 1.4.5 (Jones decomposition of A_p weights) *Let ω be nonnegative locally integrable function. Then for $1 < p < \infty$, $\omega \in A_p$ if and only if there are $\omega_1, \omega_2 \in A_1$ such that $\omega(x) = \omega_1(x) \cdot \omega_2(x)^{1-p}$.*

Proof. we only consider the necessity. Since $\omega \in A_p$, by the property (ii) $\omega(x)^{1-p'} \in A_{p'}$.

First let $1 < p \le 2$, then $s = p' - 1 > 1$. Take a nonnegative function $u_0(x) \in L^{p'}(\mathbb{R}^n, \omega dx)$, let us construct a function sequence $\{u_j(x)\}$ by the Hardy-Littlewood maximal operator M as follows.

$$u_{j+1}(x) = [M(u_j^s)(x)]^{1/s} + \omega(x)^{-1}M(u_j\omega)(x), \quad \text{for} \quad j = 0, 1, \cdots.$$

Then we have the following two conclusions.

(a) There exists a constant C such that $\|u_{j+1}\|_{p',\,\omega dx} \le C\|u_j\|_{p',\,\omega dx}$, for $j = 0, 1, \cdots$.

(b) For the constant C appearing in (a), take $\delta > C$ and denote

$$U(x) = \sum_{j=0}^{\infty} \delta^{-j} u_j(x),$$

then $U\omega \in A_1$ and $U^s \in A_1$.

In fact, by Theorem 1.4.3, we have that

$$\begin{aligned}
&\|u_{j+1}\|_{p',\,\omega dx}^{p'}\\
&\le C\Big(\int_{\mathbb{R}^n}[M(u_j^s)(x)]^{p'/s}\omega(x)dx+\int_{\mathbb{R}^n}[\omega(x)^{-1}M(u_j\omega)(x)]^{p'}\omega(x)dx\Big)\\
&= C\Big(\int_{\mathbb{R}^n}[M(u_j^s)(x)]^{p}\omega(x)dx+\int_{\mathbb{R}^n}[M(u_j\omega)(x)]^{p'}\omega(x)^{1-p'}dx\Big)\\
&\le C_1\Big(\int_{\mathbb{R}^n}u_j^{p'}(x)\omega(x)dx+\int_{\mathbb{R}^n}[u_j(x)\omega(x)]^{p'}\omega(x)^{1-p'}dx\Big)\\
&\le C\int_{\mathbb{R}^n}u_j^{p'}(x)\omega(x)dx.
\end{aligned}$$

So the conclusion (a) holds and it is easy to check that the constant C is independent of u_j for all $j \ge 0$. Now let us verify (b). By the definition of u_{j+1}, we have that

$$\begin{aligned}
M(U\omega)(x) &\le \sum_{j=0}^{\infty}\delta^{-j}M(u_j\omega)(x)\\
&\le \sum_{j=0}^{\infty}\delta^{-j}u_{j+1}(x)\omega(x)\\
&= \delta\sum_{j=1}^{\infty}\delta^{-j}u_j(x)\omega(x)\\
&\le \delta(U(x)\omega(x)).
\end{aligned}$$

Hence $U\omega \in A_1$. On the other hand, we have that

$$[M(U^s)(x)]^{1/s} \le \sum_{j=0}^{\infty}\delta^{-j}[M(u_j^s)(x)]^{1/s} \le \sum_{j=0}^{\infty}\delta^{-j}u_{j+1}(x) \le \delta U(x).$$

So $U^s \in A_1$ too. Thus, if denote $\omega_1 = U\omega$ and $\omega_2 = U^s$, then $\omega_1, \omega_2 \in A_1$ and

$$\omega = (U\omega)\cdot(U^s)^{-1/s} = (U\omega)\cdot(U^s)^{1-p} = \omega_1\cdot\omega_2^{1-p}.$$

Hence the necessity of theorem holds for $1 < p \le 2$.

If $2 < p < \infty$ and $\omega \in A_p$, then $\omega^{1-p'} \in A_{p'}$ and $1 < p' \le 2$. Thus by the above proof process, there are $\nu_1, \nu_2 \in A_1$ such that $\omega^{1-p'} = \nu_1 \cdot \nu_2^{1-p'}$. Equivalently, $\omega = \nu_2 \cdot \nu_1^{1-p}$. Thus we complete the proof of Theorem 1.4.5. ■

Below we give a sharp result about A_1 weights.

Proposition 1.4.3 *Let $x \in \mathbb{R}^n$. Then $|x|^\alpha \in A_1$ if and only if $-n < \alpha \le 0$.*

Proof. If $|x|^\alpha \in A_1$. Then $|x|^\alpha \in L^1_{loc}(\mathbb{R}^n)$, hence the condition $\alpha > -n$ is necessary. On the other hand, if $\alpha > 0$ and $|x|^\alpha \in A_1$, we take $1 < p \le 1 + \alpha/n$, then $|x|^\alpha \in A_p$ by the property (i) of A_p weights. However, $|x|^{\alpha(1-p')} \notin L^1_{loc}(\mathbb{R}^n)$ for the choice of p.

Conversely, suppose that $-n < \alpha \le 0$, we will show that $|x|^\alpha \in A_1$. Indeed, for any fixed cube Q, we denote Q_0 is the translation of Q with the center at origin.

The case (i): $2Q_0 \cap Q \ne \emptyset$. In this case, we have $4Q_0 \supset Q$, and

$$\frac{1}{|Q|}\int_Q |x|^\alpha dx \le \frac{1}{|Q|}\int_{4Q_0} |x|^\alpha dx \le C|Q|^{\alpha/n} \le C \inf_{x\in Q} |x|^\alpha.$$

The case (ii): $2Q_0 \cap Q = \emptyset$. Notice that if $x, y \in Q$, then

$$|x| \le |x-y| + |y| \le C|Q|^{1/n} + |y| \le (C+1)|y|,$$

Thus

$$\frac{1}{|Q|}\int_Q |x|^\alpha dx \le C \inf_{x\in Q} |x|^\alpha.$$

It is easy to see that the constants C in the cases both (i) and (ii) depend only on n. Therefore, we have proved that $|x|^\alpha \in A_1$. ∎

By the property (iii) of A_p weights, we get immediately the following sharp result for A_p weights.

Proposition 1.4.4 *Let $x \in \mathbb{R}^n$. Then for $1 < p < \infty$, $|x|^\alpha \in A_p$ if and only if $-n < \alpha < n(p-1)$.*

1.5 Notes and references

Theorem 1.1.1 was first proved by Hardy and Littlewood [HaL] for $n = 1$ and then by Wiener [Wi] for $n > 1$. The idea of proof given here was taken from Stein [St4] which is one of the most important monograph in harmonic analysis.

The Calderón-Zygmund decomposition (Theorems 1.2.1 and 1.2.3) first appeared in Calderón and Zygmund [CaZ1] which is regarded as the foundation of several variables singular integrals theory.

Theorem 1.2.2 is due to Stein [St2]. The idea of proof given here was taken from Garcia-Cuerva and Rubia de Francia [GaR], which is a nice monograph on the topic of weighted norm inequalities.

The Marcinkiewicz interpolation theorem was first announced by Marcinkiewicz [Ma]. Its complete proof can be found in Zygmund [Zy]. Theorem 1.3.1 in this chapter is general form of the Marcinkiewicz interpolation theorem, whose proof was taken from Duoandikoetxea [Du2].

Theorem 1.3.2 is due to Fefferman and Stein [FeS1].

The Reverse Hölder inequality (Theorem 1.4.1) first was proved by Coifman and Fefferman [CoiF].

Theorem 1.4.2 and Theorem 1.4.3 were first proved by Muckenhoupt [Mu] for $n = 1$. So, the A_p weights are also called as Muckenhoupt A_p weights. Precisely, a function ω is called to satisfy Muckenhoupt A_p condition if $\omega \in A_p$. The idea of proving Theorems 1.4.2 and 1.4.3 given here was taken from Journé [Jou].

Theorem 1.4.4 is due to Coifman and Rochberg [CoiR].

Theorem 1.4.5 was first proved by Jones [Jon]. The proof of Theorems 1.4.5 given here was taken from Coifman, Jones and Rubia de Francia [CoiJR].

Chapter 2

Singular Integral Operators

Calderón-Zygmund singular integral operator is a direct generalization of the Hilbert transform and the Riesz transform. The former is originated from researches of boundary value of conjugate harmonic functions on the upper half-plane, and the latter is tightly associated to the regularity of solution of second order elliptic equation.

Now we will introduce their backgrounds briefly.

Suppose $f \in L^p(\mathbb{R})$ $(1 \leq p < \infty)$. Consider the Cauchy integral on $\mathbb{R}$:

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} dt,$$

where $z = x + iy, y > 0$. It is clear to see that $F(z)$ is analytic on $\mathbb{R}^2_+$. Note that

$$\begin{aligned} F(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t) dt + \frac{i}{2\pi} \int_{\mathbb{R}} \frac{x-t}{(x-t)^2+y^2} f(t) dt \\ &:= \frac{1}{2} \Big[(P_y * f)(x) + i(Q_y * f)(x) \Big], \end{aligned}$$

where

$$P_y(t) = \frac{1}{\pi} \frac{y}{t^2 + y^2}$$

is called the Poisson kernel, and

$$Q_y(t) = \frac{1}{\pi} \frac{t}{t^2 + y^2}$$

is called the conjugate Poisson kernel. Correspondingly, $P_y * f$ is called the Poisson integral of f, and $Q_y * f$ is called the conjugate Poisson integral of f. From the property of boundary values of harmonic functions, it follows that $P_y * f \to f$, a.e. as $y \to 0$ and $Q_y * f \to Hf$, a.e. as $y \to 0$. Hf is called the Hilbert transform of f.

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt. \tag{2.0.1}$$

It can be proved that if $f \in L^2(\mathbb{R})$, then

$$\widehat{Hf}(\xi) = -i\text{sgn}\xi \hat{f}(\xi). \tag{2.0.2}$$

Let $K(x) = \text{p.v.}\frac{1}{x}$, then $Hf = (K * f)$.

Now suppose that $f \in L^2(\mathbb{R}^n)$ and give the Poisson equation $\triangle u = f$, where

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian operator on $\mathbb{R}^n$. By taking the Fourier transform on both sides of the equation we have that

$$\hat{f}(\xi) = -4\pi^2|\xi|^2\hat{u}(\xi).$$

This is equivalent to

$$\hat{u}(\xi) = -\frac{1}{4\pi^2|\xi|^2}\hat{f}(\xi).$$

Thus, for $1 \leq j, k \leq n$,

$$(\widehat{\frac{\partial^2 u}{\partial x_j \partial x_k}})(\xi) = -4\pi^2\xi_j\xi_k\hat{u}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}\hat{f}(\xi).$$

If we define the operator

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi), \qquad j = 1, 2, \cdots, n, \tag{2.0.3}$$

then it implies that

$$(\widehat{\frac{\partial^2 u}{\partial x_j \partial x_k}})(\xi) = -\widehat{R_j R_k f}(\xi).$$

The operator R_j defined by (2.0.3) is called the Riesz transform. Thus the problem of regularity of solution for the Poisson equation is transfered to that of boundedness of the Riesz transform. By comparing (2.0.2) with (2.0.3), it is evident to see that the Riesz transform is a generalization of the Hilbert transform from one dimension to n dimension. If $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, then the Riesz transform of f has the following form:

$$R_j f(x) = \text{p.v.} C_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy, \quad 1 \le j \le n. \tag{2.0.4}$$

If we let $K_j(x) = \text{p.v.} \frac{x_j}{|x|^{n+1}}$ $(j = 1, 2, \cdots, n)$, then

$$R_j f = K_j * f.$$

Let us now see another background of singular integral operator. We know, when $n \ge 3$, the basic solution of the Laplacian operator $\triangle$ is

$$\Gamma(x) = \frac{1}{(2-n)\omega_{n-1}} \frac{1}{|x|^{n-2}}.$$

Thus when f has good properties, for example $f \in \mathscr{S}(\mathbb{R}^n)$, $\Gamma * f$ is a solution of the Poisson equation $\triangle u = f$, that is

$$u(x) = \Gamma * f(x) = C_n \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy.$$

Formally, by taking partial derivatives of second order of u, we obtain that

$$\frac{\partial^2 u(x)}{\partial x_j^2} = \int_{\mathbb{R}^n} \frac{\Omega_j(x-y)}{|x-y|^n} f(y)dy := \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{\Omega_j(x-y)}{|x-y|^n} f(y)dy,$$

where $\Omega_j(y) = C_n(1 - n|y|^{-2} y_j^2)$. It is not difficult to prove that Ω_j satisfies the following properties:

(a) $\Omega_j(\lambda y) = \Omega_j(y), \ \forall \lambda > 0$;

(b) $\int_{\mathbb{S}^{n-1}} \Omega_j(y') d\sigma(y') = 0$;

(c) $\Omega_j \in L^1(\mathbb{S}^{n-1})$.

Put $T_j f(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{\Omega_j(x-y)}{|x-y|^n} f(y)dy$, then L^p regularity of solution of equation $\triangle u = f$ is converted to the L^p boundedness of operator T_j.

Suppose that a function Ω satisfies above three conditions (a), (b) and (c), then for $f \in L^p(\mathbb{R}^n)(1 \le p < \infty)$, define an integral operator T_Ω by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy. \tag{2.0.5}$$

If we take $\Omega(x) = \dfrac{x_j}{|x|}$, then T_Ω becomes the Riesz transform $R_j(j = 1, 2, \cdots, n)$. If we take $n = 1$ and $\Omega(x) = \text{sgn}x$, then T_Ω is just the Hilbert transform H. The operator T_Ω defined by (2.0.5) is the object of study in this chapter.

2.1 Calderón-Zygmund singular integral operators

Definition 2.1.1 *Suppose that $K(x) \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$ and satisfies the following conditions:*

$$|K(x)| \le B|x|^{-n}, \qquad \forall\, x \neq 0; \tag{2.1.1}$$

$$\int_{r \le |x| \le R} K(x)dx = 0, \qquad \forall\, 0 < r < R < \infty; \tag{2.1.2}$$

$$\int_{|x| \ge 2|y|} |K(x-y) - K(x)|dx \le B, \qquad \forall\, y \neq 0. \tag{2.1.3}$$

Then K is called the Calderón-Zygmund kernel, where B is a constant independent of x and y. And the condition (2.1.3) is called as Hömander's condition.

Theorem 2.1.1 *Suppose that K is the Calderón-Zygmund kernel. For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n)(1 < p < \infty)$, let*

$$T_\varepsilon f(x) = \int_{|y| \ge \varepsilon} f(x-y)K(y)dy,$$

then the following statements hold.

(i) $\|T_\varepsilon f\|_p \le A_p \|f\|_p$, where A_p is independent of ε and f.

(ii) For any $f \in L^p(\mathbb{R}^n)$, $\lim\limits_{\varepsilon \to 0} T_\varepsilon f$ exists in the sense of L^p norm. That is, these exists a T such that

$$Tf(x) = p.v. \int_{\mathbb{R}^n} f(x-y)K(y)dy.$$

(iii) $\|Tf\|_p \le A_p \|f\|_p$.

Remark 2.1.1 *The linear operator T defined by Theorem 2.1.1 (ii) is called the Calderón-Zygmund singular integral operator. T_ε is also called the truncated operator of T.*

Proof. For $\varepsilon > 0$, let $K_\varepsilon(x) = K(x)\chi_{\{|x|\geq\varepsilon\}}(x)$, then $T_\varepsilon f(x) = K_\varepsilon * f(x)$. First we will prove that T_ε is of type $(2,2)$, and $\{T_\varepsilon\}$ is uniformly bounded on $L^2(\mathbb{R}^n)$. It is clear to check that K_ε satisfies (2.1.1) and (2.1.2) uniformly in ε. Next we will show that K_ε also satisfies (2.1.3) uniformly. Actually, for any $x, y \in \mathbb{R}^n$ with $y \neq 0$ and $|x| \geq 2|y|$, if both x and $x - y$ are in $B(0,\varepsilon)$, then $K_\varepsilon(x) = K_\varepsilon(x-y) = 0$; If both x and $x-y$ are in $(B(0,\varepsilon))^c$, then $K_\varepsilon(x) = K(x), K_\varepsilon(x-y) = K(x-y)$. In this case, K_ε satisfies (2.1.3). If $|x| > \varepsilon$ and $|x-y| < \varepsilon$, then we have $|x|/2 \leq |x-y| < \varepsilon$ and $\varepsilon < |x| < 2\varepsilon$. Therefore, we have that

$$\int_{|x|\geq 2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)|dx \leq \int_{\varepsilon\leq|x|\leq 2\varepsilon} |K_\varepsilon(x)|dx \leq CB,$$

where C is independent of ε. By similar way we can prove, when $|x| < \varepsilon$ and $|x-y| > \varepsilon$, K_ε satisfies (2.1.3) uniformly in ε.

Next we will show that $\{T_\varepsilon\}$ is uniformly bounded in ε on $L^2(\mathbb{R}^n)$. Actually, for any $\varepsilon > 0$, $K_\varepsilon \in L^2(\mathbb{R}^n)$, so by the Plancherel Theorem, it suffices to prove that there exists a constant $C > 0$ such that, for any $\varepsilon > 0$,

$$\sup_{\xi\in\mathbb{R}^n} |\hat{K_\varepsilon}(\xi)| \leq CB. \tag{2.1.4}$$

In fact, for $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
\hat{K_\varepsilon}(\xi) &= \lim_{R\to\infty} \int_{|x|\leq R} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)dx \\
&= \lim_{R\to\infty} \left(\int_{|x|\leq\frac{a}{|\xi|}} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)dx + \int_{\frac{a}{|\xi|}<|x|\leq R} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)dx \right) \\
&:= \lim_{R\to\infty} (I_1 + I_2).
\end{aligned}$$

From condition (2.1.1) and (2.1.2), it follows that

$$\begin{aligned}
|I_1| &= \left| \int_{|x|\leq\frac{a}{|\xi|}} (e^{-2\pi i x\cdot\xi} - 1) K_\varepsilon(x)dx \right| \\
&\leq C|\xi| \int_{|x|\leq\frac{a}{|\xi|}} |x||K_\varepsilon(x)|dx \\
&\leq CaB.
\end{aligned}$$

Now consider I_2. Take $y = \frac{\xi}{2|\xi|^2}$, then $e^{2\pi i y\cdot\xi} = -1$. Thus

$$\begin{aligned} I_2 &= \int_{\frac{a}{|\xi|}<|x-y|\le R} e^{-2\pi i(x-y)\cdot\xi} K_\varepsilon(x-y)dx \\ &= -\int_{\frac{a}{|\xi|}<|x-y|\le R} e^{-2\pi i x\cdot\xi} K_\varepsilon(x-y)dx \\ &= -\int_{\frac{a}{|\xi|}<|x|\le R} e^{-2\pi i x\cdot\xi} K_\varepsilon(x-y)dx + J, \end{aligned}$$

where

$$J = \left(\int_{\frac{a}{|\xi|}<|x|\le R} - \int_{\frac{a}{|\xi|}<|x-y|\le R}\right) e^{-2\pi i x\cdot\xi} K_\varepsilon(x-y)dx.$$

Hence we have

$$I_2 = \frac{1}{2}\int_{\frac{a}{|\xi|}<|x|\le R} [K_\varepsilon(x) - K_\varepsilon(x-y)]e^{-2\pi i x\cdot\xi}dx + \frac{J}{2}.$$

As $|y| = \frac{1}{2|\xi|}$ and $a > 1$, we have that

$$\begin{aligned} &\left|\int_{\frac{a}{|\xi|}<|x|\le R} [K_\varepsilon(x) - K_\varepsilon(x-y)]e^{-2\pi i x\cdot\xi}dx\right| \\ \le &\int_{|x|\ge 2|y|} |K_\varepsilon(x) - K_\varepsilon(x-y)|dx \le CB, \end{aligned}$$

where C, B are independent of ε, ξ.

On the other hand, if we let E be the symmetric difference of both sets $\{x : \frac{a}{|\xi|} < |x| \le R\}$ and $\{x : \frac{a}{|\xi|} < |x-y| \le R\}$, then

$$|J| \le \int_E |K_\varepsilon(x-y)|dx.$$

Noting that $|y| = \frac{1}{2|\xi|}$ and $a > 1$, it follows that

$$E \subset \left\{x : \frac{a}{2|\xi|} \le |x| \le \frac{2a}{|\xi|}\right\} \bigcup \left\{x : \frac{R}{2} \le |x| \le 2R\right\}.$$

Thus by (2.1.1), it implies that

$$\begin{aligned} |J| &\le \int_{\frac{a}{2|\xi|}\le|x|\le\frac{2a}{|\xi|}} |K_\varepsilon(x-y)|dx + \int_{\frac{R}{2}\le|x|\le 2R} |K_\varepsilon(x-y)|dx \\ &\le CB, \end{aligned}$$

where C, B are independent of ξ, ε. Sum up all above we get (2.1.4), therefore $\{T_\varepsilon\}$ is uniformly bounded in ε on $L^2(\mathbb{R}^n)$.

Now we show that T_ε is of weak type (1, 1), and the bound is independent of ε.

For every $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, by the Calderón-Zygmund decompositin of a function (Theorem 1.2.3) we can get a series of non-overlapping cubes $\{Q_j\}$ and two functions g, b, such that $f = g + b$ which satisfy the following properties:

(a) $\|g\|_2^2 \le C\lambda \|f\|_1, \quad |g(x)| \le 2^n \lambda, \quad a.e.\ x \in \mathbb{R}^n;$

(b) $\lambda \le \dfrac{1}{|Q_j|} \displaystyle\int_{Q_j} |f(x)|dx \le 2^n \lambda$, for every Q_j;

(c) $\displaystyle\sum_j |Q_j| \le \frac{1}{\lambda}\|f\|_1;$

(d) $b(x) = \displaystyle\sum_j b_j(x), \int_{Q_j} b_j(x)dx = 0, \mathrm{supp} b_j \subset Q_j$ and $\|b_j\|_1 \le 2 \displaystyle\int_{Q_j} |f(x)|dx.$

Since $T_\varepsilon f(x) = T_\varepsilon g(x) + T_\varepsilon b(x)$, we have that

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : |T_\varepsilon f(x)| > \lambda\}| \\
&\quad \le \left|\left\{x \in \mathbb{R}^n : |T_\varepsilon g(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : |T_\varepsilon b(x)| > \frac{\lambda}{2}\right\}\right| \\
&\quad := I_1 + I_2.
\end{aligned}$$

The first step and (a) imply that

$$I_1 \le \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}^n} |T_\varepsilon g(x)|^2 dx \le \frac{4C}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^2 dx \le \frac{4C}{\lambda}\|f\|_1.$$

To estimate I_2, let $Q_j^* = 2\sqrt{n} Q_j$ be a cube whose center is the same as Q_j and side is $2\sqrt{n}$ times that of Q_j. Let $E^* = \bigcup_j Q_j^*$, then (c) implies that

$$|E^*| \le \sum_j |Q_j^*| \le \frac{C_n}{\lambda}\|f\|_1.$$

Thus

$$\begin{aligned}
I_2 &\le |E^*| + \left|\left\{x \notin E^* : |T_\varepsilon b(x)| > \frac{\lambda}{2}\right\}\right| \\
&\le \frac{C_n}{\lambda}\|f\|_1 + \frac{2}{\lambda}\int_{\mathbb{R}^n \setminus E^*} |T_\varepsilon b(x)| dx.
\end{aligned}$$

Notice that $|T_\varepsilon b(x)| \le \sum_j |T_\varepsilon b_j(x)|$, it suffices to prove that

$$\sum_j \int_{\mathbb{R}^n \setminus E^*} |T_\varepsilon b_j(x)| dx \le C\|f\|_1. \tag{2.1.5}$$

Denote the center of Q_j by y_j, then by (2.1.3) we have that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus E^*} |T_\varepsilon b_j(x)|dx &\le \int_{\mathbb{R}^n\setminus Q_j^*}\int_{Q_j} |K_\varepsilon(x-y)-K_\varepsilon(x-y_j)||b_j(y)|dydx\\ &\le \int_{Q_j}|b_j(y)|\int_{\mathbb{R}^n\setminus Q_j^*}|K_\varepsilon(x-y)-K_\varepsilon(x-y_j)|dxdy\\ &\le CB\int_{Q_j}|b_j(y)|dy \le 2CB\int_{Q_j}|f(y)|dy,\end{aligned}$$

which, together with (b) and (c), yields (2.1.5). It follows that T_ε is of weak type (1, 1) and its bound is independent of ε or f.

Now we will show that T_ε is of type (p,p) $(1<p<\infty)$. Applying the Marcinkiewicz Interpolation Theorem (Theorem 1.3.1) we know that T_ε is of type (p,p) $(1<p<2)$, and its bound is independent of ε or f. Now suppose $2<p<\infty$, and $\frac{1}{p}+\frac{1}{p'}=1$, then $1<p'<2$. If we let $\widetilde{T_\varepsilon}$ be the dual operator of T_ε, then $\widetilde{T_\varepsilon}f(x)=\widetilde{K_\varepsilon}*f(x)$, where $\widetilde{K_\varepsilon}(x)=\overline{K_\varepsilon(-x)}$. Clearly, $\widetilde{K_\varepsilon}$ satisfies all the conditions of K_ε. Thus $\widetilde{T_\varepsilon}$ is of type (p',p'). Thus for any $f\in L^p(\mathbb{R}^n)$, we have that

$$\begin{aligned}\|T_\varepsilon f\|_p &= \sup_{\|g\|_{p'}\le 1}\left|\int_{\mathbb{R}^n} T_\varepsilon f(x)g(x)dx\right|\\ &= \sup_{\|g\|_{p'}\le 1}\left|\int_{\mathbb{R}^n} f(x)\widetilde{T_\varepsilon}g(x)dx\right|\\ &\le \|f\|_p\cdot\sup_{\|g\|_{p'}\le 1}\left\|\widetilde{T_\varepsilon}g\right\|_{p'}\\ &\le A_p\|f\|_p.\end{aligned}$$

Here A_p is independent of ε or f. This finishes the proof of (i).

Next we will illustrate that for any $f\in L^p(\mathbb{R}^n)$ $(1<p<\infty)$, Tf exists and is the limit of $\{T_\varepsilon f\}$ in L^p. First suppose that $f\in C_0^\infty(\mathbb{R}^n)$. For any y $(y\neq 0)$ we wish to obtain

$$\left(\int_{\mathbb{R}^n}|f(x-y)-f(x)|^p dx\right)^{\frac{1}{p}} \le C|y|. \tag{2.1.6}$$

Actually, from

$$\frac{d}{dt}f(x-ty) = \langle \nabla f, -y\rangle(x-ty),$$

it follows that

$$\begin{aligned} f(x-y)-f(x) &= \int_0^1 \frac{d}{dt}f(x-ty)dt \\ &= \int_0^1 \langle \nabla f, -y\rangle(x-ty)dt \\ &= \int_0^{|y|} \langle \nabla f, -y'\rangle(x-sy')ds, \end{aligned}$$

where $y' = \frac{y}{|y|}$. Thus

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |f(x-y)-f(x)|^p dx\right)^{\frac{1}{p}} &= \left(\int_{\mathbb{R}^n} \left|\int_0^{|y|} \langle \nabla f, -y'\rangle(x-sy')ds\right|^p dx\right)^{\frac{1}{p}} \\ &\le \int_0^{|y|} \left(\int_{\mathbb{R}^n} |\langle \nabla f, -y'\rangle(x-sy')|^p dx\right)^{\frac{1}{p}} ds \\ &\le |y| \cdot \sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_p . \end{aligned}$$

Now assume $0 < \eta < \varepsilon$, then (2.1.6) and (2.1.1) implies that

$$\begin{aligned} \|T_\eta f - T_\varepsilon f\|_p &\le \int_{\eta<|y|\le\varepsilon} |K(y)| \left(\int_{\mathbb{R}^n} |f(x-y)-f(x)|^p dx\right)^{\frac{1}{p}} dy \\ &\le C\int_{\eta<|y|\le\varepsilon} |y|\cdot|K(y)|dy \\ &\le CB\varepsilon \longrightarrow 0 \qquad (\text{as } \ \eta,\varepsilon \longrightarrow 0). \end{aligned}$$

This shows that for every function f in $C_0^\infty(\mathbb{R}^n)$, $\{T_\varepsilon f\}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$, therefore there exists $Tf \in L^p$ such that

$$\lim_{\varepsilon\to 0} \|T_\varepsilon f - Tf\|_p = 0.$$

It immediately follows that

$$\|Tf\|_p \le \|Tf - T_\varepsilon f\|_p + \|T_\varepsilon f\|_p \le \|Tf - T_\varepsilon f\|_p + A_p\|f\|_p,$$

so

$$\|Tf\|_p \le A_p\|f\|_p.$$

For any $f \in L^p(\mathbb{R}^n)$ and $\delta > 0$, there exists $g \in C_0^\infty(\mathbb{R}^n)$ such that $f = g + h$ and $\|h\|_p < \delta$. Thus, for $0 < \eta < \varepsilon$,

$$\begin{aligned}\|T_\eta f - T_\varepsilon f\|_p &\leq \|T_\eta (f-g)\|_p + \|T_\eta g - T_\varepsilon g\|_p + \|T_\varepsilon (g-f)\|_p \\ &\leq A_p\|f-g\|_p + \|T_\eta g - T_\varepsilon g\|_p + A_p\|g-f\|_p \\ &\longrightarrow 2A_p\delta, \quad \text{as} \quad \eta, \varepsilon \to 0.\end{aligned}$$

Since δ is arbitrary, we conclude that $\{T_\varepsilon f\}$ is still a Cauchy sequence in $L^p(\mathbb{R}^n)$ for any $f \in L^p(\mathbb{R}^n)$. Thus there exists $Tf \in L^p$, such that

$$\lim_{\varepsilon \to 0} \|Tf - T_\varepsilon f\|_p = 0$$

and

$$\|Tf\|_p \leq A_p \|f\|_p.$$

This completes the proof of Theorem 2.1.1. ∎

Let us now consider a dilation δ_ε in $\mathbb{R}^n$. Define $\delta_\varepsilon f(x) = f(\varepsilon x)$ for $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Suppose that $Tf = K * f$, and T commute with dilation, i.e., $T\delta_\varepsilon = \delta_\varepsilon T$. Then the kernel $K(x)$ of T satisfies:

$$K(\varepsilon x) = \varepsilon^{-n} K(x). \tag{2.1.7}$$

The formula (2.1.7) shows that K is homogeneous of degree $-n$. Thus we can rewrite $K(x)$ as $\frac{\Omega(x)}{|x|^n}$, where Ω satisfies the homogeneous condition of degree zero, i.e., $\Omega(\lambda x) = \Omega(x)$ for every $\lambda > 0$ and $x \neq 0$. In this case, due to the conditions (2.1.1) and (2.1.3), $\Omega(x')$ should satisfy:

(a) $|K(x)| \leq \frac{B}{|x|^n} \Longleftrightarrow |\Omega(x')| \leq B$, for every $x' \in S^{n-1}$;

(b) $\displaystyle\int_{r<|x|\leq R} K(x)dx = 0 \Longleftrightarrow \int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0$; where σ is a measure on $\mathbb{S}^{n-1}$ induced by the Lebesgue measure.

(c) The condition (2.1.3) will be changed to a stronger Dini's condition:

$$\int_0^1 \frac{\omega_\infty(\delta)}{\delta} d\delta < \infty,$$

where

$$\omega_\infty(\delta) = \sup_{x', y' \in S^{n-1}, |x'-y'|<\delta} |\Omega(x') - \Omega(y')|.$$

The condition (b) is derived from the following equality.

$$\int_{r<|x|\leq R} K(x)dx = \int_r^\infty \int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') \frac{dr}{r} = \log \frac{R}{r} \int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x').$$

Remark 2.1.2 *The condition mentioned above is called L^∞-Dini condition.*

Theorem 2.1.2 *Suppose that $\Omega(x)$ is a homogeneous bounded function of degree 0 on $\mathbb{R}^n$, with*

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0 \tag{2.1.8}$$

and

$$\int_0^1 \frac{\omega_\infty(\delta)}{\delta} d\delta < \infty. \tag{2.1.9}$$

Let

$$T_\varepsilon f(x) = \int_{|y|\geq\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy$$

for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then the following three statements hold:

(i) there exists a constant A_p independent of f, ε such that $\|T_\varepsilon f\|_p \leq A_p\|f\|_p$;
(ii) there exists Tf such that $\lim\limits_{\varepsilon\to 0} T_\varepsilon f(x) = Tf(x)$ in L^p norm;
(iii) $\|Tf\|_p \leq A_p\|f\|_p$.

Proof. By Theorem 2.1.1, we merely need to show that $K(x) = \frac{\Omega(x)}{|x|^n}$ satisfies the condition (2.1.3).

$$K(x-y) - K(x) = \frac{\Omega(x-y) - \Omega(x)}{|x-y|^n} + \Omega(x)\left(\frac{1}{|x-y|^n} - \frac{1}{|x|^n}\right) := I_1 + I_2.$$

when $|x| \geq 2|y|$ with $0 < \theta < 1$, we have that

$$|x - \theta y| \leq |x| + |y| \leq \frac{3}{2}|x|$$

and

$$|x-y| \geq |x| - |y| \geq \frac{1}{2}|x|.$$

Thus

$$\left|\frac{1}{|x-y|^n} - \frac{1}{|x|^n}\right| \leq C\frac{|y||x-\theta y|^{n-1}}{|x-y|^n|x|^n} \leq C\frac{|y|}{|x|^{n+1}}. \tag{2.1.10}$$

On the other hand, when $|x| \geq 2|y|$,

$$\left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right| \leq 2\frac{|y|}{|x|}. \tag{2.1.11}$$

Therefore from (2.1.10) and (2.1.11) it follows that

$$\begin{aligned}
\int_{|x|\geq 2|y|} |K(x-y)-K(x)|dx &\leq \int_{|x|\geq 2|y|} |I_1|dx + \int_{|x|\geq 2|y|} |I_2|dx \\
&\leq \int_{|x|\geq 2|y|} \left|\Omega\left(\frac{x-y}{|x-y|}\right) - \Omega\left(\frac{x}{|x|}\right)\right| \frac{dx}{|x-y|^n} \\
&\qquad + C\|\Omega\|_\infty |y| \int_{|x|\geq 2|y|} \frac{dx}{|x|^{n+1}} \\
&\leq C\int_{|x|\geq 2|y|} \omega_\infty\left(2\frac{|y|}{|x|}\right)\frac{dx}{|x|^n} + C'\|\Omega\|_\infty \\
&= C\int_{2|y|}^\infty \int_{\mathbb{S}^{n-1}} \omega_\infty\left(2\frac{|y|}{r}\right) d\sigma(x')\frac{dr}{r} + C'\|\Omega\|_\infty \\
&\leq C'\int_0^1 \frac{\omega_\infty(\delta)}{\delta} d\delta + C\|\Omega\|_\infty \\
&\leq B.
\end{aligned}$$

■

Remark 2.1.3 *If* $K(x) = \frac{\Omega(x)}{|x|^n}$, *where* Ω *is homogeneous of degree zero, then* T_Ω *defined by*

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)dy \tag{2.1.12}$$

is also called a singular integral operator with homogeneous kernel.

Both Theorem 2.1.1 and Theorem 2.1.2 show that the L^p-norm limit of Calderón-Zygmund singular integral operator and singular integral operator with homogeneous kernel exist while considering them as the truncated operator family, and they are both operators of type $(p,p)(1<p<\infty)$. A natural question is whether the limit of $\{T_\varepsilon f(x)\}$ in pointwise sense exist for any $f \in L^p(\mathbb{R}^n)(1 \leq p < \infty)$. In the following we will give an affirmative answer by introducing the maximal operator of singular integral operator, meanwhile we will show the weak (1, 1) boundedness of T_Ω.

Suppose that T_Ω is a singular integral operator defined by (2.1.12). Let Ω be a homogeneous of degree 0 on $\mathbb{R}^n$ and satisfy (2.1.8) and (2.1.9). For $f \in L^p(\mathbb{R}^n)(1 \leq p < \infty)$,

$$T_\Omega^* f(x) = \sup_{\varepsilon>0} |T_{\Omega,\varepsilon} f(x)|$$

is called the maximal singular integral operator, where $T_{\Omega,\varepsilon}$ is the truncated operator of T_Ω defined by

$$T_{\Omega,\varepsilon} f(x) = \int_{|y|\geq\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy, \qquad \varepsilon > 0.$$

Next we will formulate a pointwise inequality of T^*_Ω.

Lemma 2.1.1 (Cotlar inequality) *There exist two constant $C_1, c_2 > 0$ such that*

$$T^*_\Omega f(x) \leq C_1 M(T_\Omega f)(x) + C_2 Mf(x), \tag{2.1.13}$$

for every $f \in L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ and $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal operator.

Proof. Let $K_\varepsilon(x) = |x|^{-n}\Omega(x)\chi_{\{|x|\geq\varepsilon\}}(x)$. Choose nonnegative, compact supported, radial function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp}(\varphi) \subset \{x : |x| \leq 1\}$ and

$$\int_{\mathbb{R}^n} \varphi(x)dx = 1.$$

Without loss of generality, we may assume that $\varphi(|x|)$ decrease in $|x|$. Thus

$$\lim_{\varepsilon\to 0} K_\varepsilon * \varphi = K * \varphi$$

holds pointwisely.

Now denote $\Phi(x) = K*\varphi(x) - K_1(x)$. Since K is homogeneous of degree $-n$, we have for $\varepsilon > 0$ that

$$\Phi_\varepsilon(x) = \varphi_\varepsilon * K(x) - K_\varepsilon(x), \tag{2.1.14}$$

where

$$\Phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\Phi\left(\frac{x}{\varepsilon}\right) \text{ and } \varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right).$$

By (2.1.14), for any $f \in L^p(\mathbb{R}^n)$, we have

$$T_{\Omega,\varepsilon} f(x) = K_\varepsilon * f(x) = (\varphi_\varepsilon * K) * f(x) - \Phi_\varepsilon * f(x). \tag{2.1.15}$$

On the other hand, for every $\eta > 0$ and $x \in \mathbb{R}^n$, it follows that

$$(\varphi_\varepsilon * K_\eta) * f(x) = \varphi_\varepsilon * (K_\eta * f)(x) = \varphi_\varepsilon * (T_{\Omega,\eta} f)(x).$$

Assume $\varphi_\varepsilon \in L^{p'}$. Then $\varphi_\varepsilon * K_\eta$ converges to $\varphi_\varepsilon * K$ in L^p as $\eta \to 0$, (see the proof of Theorem 2.1.1), meanwhile $T_{\Omega,\eta} f$ converges to $T_\Omega f$ in L^∞ . Thus

$$(\varphi_\varepsilon * K) * f(x) = \varphi_\varepsilon * (T_\Omega f)(x).$$

This formula together with (2.1.15) implies that

$$T_{\Omega,\varepsilon}f(x) = \varphi_\varepsilon * (T_\Omega f)(x) - \Phi_\varepsilon * f(x). \tag{2.1.16}$$

Next we will prove that Φ can be dominated by a radial integrable function. When $|x| < 1$,

$$\Phi(x) = \varphi * K(x) = \int_{\mathbb{R}^n} [\varphi(x-y) - \varphi(x)]K(y)dy.$$

Note that $K(y) = |y|^{-n}\Omega(y)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\mathrm{supp}\varphi \subset \{x : |x| \le 1\}$. Clearly Φ is bounded when $|x| < 1$. Also, when $1 \le |x| \le 2$, Φ is still bounded. When $|x| > 2$, we have that

$$\begin{aligned}\Phi(x) &= \int_{\mathbb{R}^n} K(x-y)\varphi(y)dy - K(x)\\ &= \int_{|y|\le 1} [K(x-y) - K(x)]\varphi(y)dy.\end{aligned}$$

Since

$$|K(x-y) - K(x)| \le \frac{|\Omega(x-y) - \Omega(x)|}{|x-y|^n} + |\Omega(x)| \left| \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right|,$$

The proof of Theorem 2.1.2 implies that

$$|K(x-y) - K(x)| \le C'|x|^{-n} \cdot \omega_\infty\left(\frac{2}{|x|}\right).$$

Thus when $|x| > 2, |\Phi(x)| \le C'|x|^{-n}\omega_\infty\left(\frac{2}{|x|}\right)$. Since Ω satisfies (2.1.10), the minimum radical dominated function of Φ, denoted by

$$\Psi(x) = \sup_{|y|\ge|x|} |\Phi(y)|,$$

must be integrable. Thus by (2.1.16) and the properties of the Hardy-Littlewood maximal operator we obtain that there exist $C_1, C_2 > 0$, such that

$$\begin{aligned}T_\Omega^* f(x) &= \sup_{\varepsilon>0} |T_{\Omega,\varepsilon}f(x)|\\ &\le \sup_{\varepsilon>0} |\varphi_\varepsilon * (T_\Omega f)(x)| + \sup_{\varepsilon>0} |\Phi_\varepsilon * f(x)|\\ &\le C_1 M(T_\Omega f)(x) + C_2 Mf(x).\end{aligned}$$

This implies (2.1.13). ∎

Since M and T_Ω are both operators of (p,p) type (see Theorem 1.1.1 and Theorem 2.1.2), it immediately follows that T^*_Ω is of type (p,p).

Theorem 2.1.3 *Suppose that Ω is a homogeneous bounded function of degree 0 and satisfies (2.1.8) and (2.1.9). Then T^*_Ω is of type (p,p) $(1<p<\infty)$ and weak type $(1,1)$.*

Proof. It suffices to show that T^*_Ω is of weak type (1, 1). The idea in the proof is the same as that of Theorem 2.1.1. For any $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, using the Calderón-Zygmund decomposition, we have $f = g + b$ and a sequence of non-overlapping cubes $\{Q_j\}$. Thus

$$|\{x : T^*_\Omega f(x) > \lambda\}| \le \left|\left\{x : T^*_\Omega g(x) > \frac{\lambda}{2}\right\}\right| + \left|\left\{x : T^*_\Omega b(x) > \frac{\lambda}{2}\right\}\right|. \quad (2.1.17)$$

Since $\|g\|_2^2 \le C\lambda\|f\|_1$ and T^*_Ω is of type $(2,2)$, we have

$$\left|\left\{x : T^*_\Omega g(x) > \frac{\lambda}{2}\right\}\right| \le C'\lambda^{-2}\|T^*_\Omega g\|_2^2 \le C''\frac{1}{\lambda}\|f\|_1.$$

Now denote the center of Q_j by y_j and the side-length of Q_j by d_j. Let $S_j = \sqrt{n}Q_j$ and $E = \bigcup_j S_j$. Thus

$$|E| \le \sum_j |S_j| = \sum_j C_n|Q_j| \le \frac{C_n}{\lambda}\|f\|_1.$$

It follows that

$$\left|\left\{x : T^*_\Omega b(x) > \frac{\lambda}{2}\right\}\right| \le |E| + \left|\left\{x \in E^c : T^*_\Omega b(x) > \frac{\lambda}{2}\right\}\right|. \quad (2.1.18)$$

Fix $x \in E^c$ and $\varepsilon > 0$, then

$$T_{\Omega,\varepsilon} b(x) = \sum_j \int_{Q_j} K_\varepsilon(x-y)b(y)dy.$$

We consider the following three cases of Q_j:

(i) for any $y \in Q_j$, $|x-y| < \varepsilon$;
(ii) for any $y \in Q_j$, $|x-y| > \varepsilon$;

(iii) there exists $y \in Q_j$, such that $|x-y| = \varepsilon$.

For the first case, $K_\varepsilon(x-y) = 0$, thus $T_{\Omega,\varepsilon}b(x) = 0$. For the second case, $K_\varepsilon(x-y) = K(x-y)$, thus

$$\left|\int_{Q_j} K_\varepsilon(x-y)b_j(y)dy\right| = \left|\int_{Q_j} [K(x-y) - K(x-y_j)]b_j(y)dy\right| \\ \leq \int_{Q_j} |K(x-y) - K(x-y_j)||b_j(y)|dy.$$

As to the third case, notice that $x \in E^c \subset S_j^c$, there exist two constants C_n and C_n' only depending on n such that $Q_j \subset S(x,r)$, where $S(x,r)$ is a closed ball with the center at x and radius $r = C_n\varepsilon$. If $y \in Q_j$, then $|x-y| \geq C_n'\varepsilon$. Thus, for $y \in Q_j$,

$$|K_\varepsilon(x-y)| \leq \frac{|\Omega(x-y)|}{|x-y|^n} \leq \|\Omega\|_\infty (C_n'\varepsilon)^{-n}.$$

Therefore

$$\begin{aligned}\left|\int_{Q_j} K_\varepsilon(x-y)b_j(y)dy\right| &\leq \int_{Q_j\cap S(x,r)} |K_\varepsilon(x-y)||b(y)|dy \\ &\leq C'\|\Omega\|_\infty \varepsilon^{-n} \int_{S(x,r)} |b(y)|dy \\ &\leq C'' \frac{1}{|S(x,r)|} \int_{S(x,r)} |b(y)|dy.\end{aligned}$$

Taking the sum of all cubes yields that

$$|T_{\Omega,\varepsilon}b(x)| \leq \sum_j \int_{Q_j} |K(x-y)-K(x-y_j)||b_j(y)|dy + \frac{C''}{|S(x,r)|}\int_{S(x,r)} |b(y)|dy.$$

Thus

$$T_\Omega^* b(x) \leq \sum_j \int_{Q_j} |K(x-y) - K(x-y_j)||b(y)|dy + CMb(x).$$

Hence

$$\begin{aligned}&\left|\left\{x \in E^c : T^*_\Omega b(x) > \frac{\lambda}{2}\right\}\right| \\ &\quad \leq \left|\left\{x \in E^c : \sum_j \int_{Q_j} |k(x-y) - K(x-y_j)||b(y)|dy > \frac{\lambda}{4}\right\}\right| \\ &\quad + \left|\left\{x \in E^c : CMb(x) > \frac{\lambda}{4}\right\}\right|.\end{aligned}$$

By (2.1.5) and the weak (1, 1) boundedness of the Hardy-Littlewood maximal operator we have

$$\left|\left\{x \in E^c : T^*_\Omega b(x) > \frac{\lambda}{2}\right\}\right| \leq \frac{C'}{\lambda}\|f\|_1.$$

This inequality together with (2.1.17) and (2.1.18) shows that T^*_Ω is operator of weak type (1, 1). ■

Corollary 2.1.1 *Suppose that Ω satisfies the conditions in Theorem 2.1.3. Then, for $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$, we have that*

$$\lim_{\varepsilon \to 0} T_{\Omega,\varepsilon} f(x) = T_\Omega f(x), \qquad a.e.\ x \in \mathbb{R}^n.$$

Proof. For $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$, let

$$\Lambda(f)(x) = \left|\lim_{\varepsilon \to 0}\sup T_{\Omega,\varepsilon} f(x) - \lim_{\varepsilon \to 0}\inf T_{\Omega,\varepsilon} f(x)\right|, \quad x \in \mathbb{R}^n,$$

then $\Lambda(f)(x) \leq 2T^*_\Omega f(x)$. For any $\delta > 0$, let $f = g + h$ such that $g \in C_0^\infty(\mathbb{R}^n)$ and $\|h\|_p < \delta$. Since Ω satisfies (2.1.8) and g is a smooth function with compact support, $T_{\Omega,\varepsilon} g$ converges to $T_\Omega g$ uniformly as $\varepsilon \to 0$. So $\Lambda(g)(x) = 0$. Then, for $1 < p < \infty$,

$$\|\Lambda(f)\|_p \leq \|\Lambda(h)\|_p \leq 2A_p\|h\|_p \leq 2A_p\delta.$$

Since δ is arbitrary, it follows that $\Lambda(f)(x) = 0$ a.e. $x \in \mathbb{R}^n$ for $1 < p < \infty$. Thus the limit of $T_{\Omega,\varepsilon} f(x)$ exists for a.e. $x \in \mathbb{R}^n$.

When $p = 1$, for any $\lambda > 0$, we also have

$$|\{x : \Lambda(f)(x) > \lambda\}| \leq \frac{2A}{\lambda}\|h\|_1 \leq \frac{2A\delta}{\lambda}.$$

Therefore we still have $\Lambda(f)(x) = 0$ a.e. $x \in \mathbb{R}^n$. Thus the limit of $T_{\Omega,\varepsilon} f(x)$ exists for a.e. $x \in \mathbb{R}^n$ with $f \in L^1(\mathbb{R}^n)$. ■

Corollary 2.1.2 *Suppose that Ω satisfies the conditions in Theorem 2.1.3. Then T_Ω is of weak type (1, 1).*

Remark 2.1.4 *The Riesz transforms R_j $(j = 1, 2, \cdots, n)$ are defined by*

$$R_j f(x) = p.v.C_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy,$$

where $C_n = \Gamma\left(\frac{n+1}{2}\right) / \pi^{\frac{n+1}{2}}$.

It is clear that the kernels $\Omega_j(x) = C_n \frac{x_j}{|x|} (j = 1, 2, \cdots, n)$ satisfies the conditions in Theorem 2.1.2. So we have

Theorem 2.1.4 *The Riesz transforms $R_j (j = 1, 2, \cdots, n)$ are of type (p, p) $(1 < p < \infty)$ and of weak type (1, 1).*

Since the Hilbert transform H is the Riesz transform of dimension 1, i.e.,

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x - y} dy,$$

of course we have

Theorem 2.1.5 *The Hilbert transform H is operator of type (p, p) $(1 < p < \infty)$ and of weak type (1, 1).*

Remark 2.1.5 *Similarly, we can also define the maximal Hilbert transform H^* and the maximal Riesz transforms R_j^*, where*

$$H^* f(x) = \sup_{\varepsilon > 0} \left| C_n \int_{|y| \geq \varepsilon} \frac{f(x - y)}{y} dy \right|,$$

$$R_j^* f(x) = \sup_{\varepsilon > 0} \left| C_n \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y) dy \right|, \qquad (j = 1, 2, \cdots, n).$$

Theorem 2.1.3 implies H^* and R_j^* $(j = 1, 2, \cdots, n)$ are all of type (p, p) $(1 < p < \infty)$ and of weak type (1, 1).

In the following we will give the weighed boundedness of the Calderón-Zygmund singular integral operator and its maximal operator. First we will formulate the definition of sharp maximal function.

Suppose $f \in L^1_{loc}(\mathbb{R}^n)$. The sharp maximal function $M^\sharp f(x)$ of f is defined by

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy, \qquad x \in \mathbb{R}^n.$$

Here the supremum is taken over all the cubes Q in $\mathbb{R}^n$ which contain x, and

$$f_Q = \frac{1}{|Q|} \int_Q f(t) dt$$

is the average of f on Q. An obvious fact is that, for every $a \in \mathbb{C}$ and $Q \subset \mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \le \frac{1}{|Q|} \int_Q |f(x) - a| dx + |a - f_Q| \le \frac{2}{|Q|} \int_Q |f(x) - a| dx. \tag{2.1.19}$$

Lemma 2.1.2 *Suppose that Ω is a homogeneous bounded function of degree 0 in $\mathbb{R}^n$ and satisfies (2.1.8) and (2.1.9). Then*

$$M^\sharp (T_\Omega f)(x) \le C(n,s) \left(M\left(|f|^s\right)(x)\right), \ x \in \mathbb{R}^n$$

holds for every $s > 1$, where M is the Hardy-Littlewood maximal operator.

Proof. For any fixed $s > 1$ and $x \in \mathbb{R}^n$, suppose Q is cube containing x, and let B be a ball of the same center and radius as Q. Let $f_1 = f\chi_{16B}$ and $f_2 = f\chi_{(16B)^c}$, then $f = f_1 + f_2$. By (2.1.19) we only need to show that

$$\frac{1}{|Q|} \int_Q |T_\Omega f(y) - T_\Omega f_2(x)| dy \le C \left(M\left(|f|^s\right)(x)\right)^{\frac{1}{s}}. \tag{2.1.20}$$

Since

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T_\Omega f(y) - T_\Omega f_2(x)| dy \\ &\le \frac{1}{|Q|} \int_Q |T_\Omega f_1(y)| dy + \frac{1}{|Q|} \int_Q |T_\Omega f_2(y) - T_\Omega f_2(x)| dy \end{aligned}$$

and $s > 1$, applying Theorem 2.1.2 we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |T_\Omega f_1(y)| dy &\le \left(\frac{1}{|Q|} \int_Q |T_\Omega f_1(y)|^s dy \right)^{\frac{1}{s}} \\ &\le C \left(\frac{1}{|Q|} \int_{16B} |f(y)|^s dy \right)^{\frac{1}{s}} \\ &\le C(n,s) \Big(M(|f|^s)(x) \Big)^{\frac{1}{s}}. \end{aligned} \tag{2.1.21}$$

On the other hand, let d be the radius of B, then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_\Omega f_2(y)-T_\Omega f_2(x)|dy\\
&=\frac{1}{|Q|}\int_Q\left|\int_{\mathbb{R}^n\setminus 16B}\left[\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(x-z)}{|x-z|^n}\right]f(z)dz\right|dy\\
&\le\frac{1}{|Q|}\int_Q\sum_{k=2}^{\infty}\int_{2^kd<|x-z|\le 2^{k+1}d}\left|\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(x-z)}{|x-z|^n}\right||f(z)|dzdy.
\end{aligned}\tag{2.1.22}$$

Note that $|x-z|\sim|y-z|$, thus

$$\left|\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(x-z)}{|x-z|^n}\right|\le C\left\{|\Omega(x-z)|\frac{|x-y|}{|x-z|^{n+1}}+\frac{|\Omega(y-z)-\Omega(x-z)|}{|x-z|^n}\right\}.$$

And when $2^kd<|x-z|\le 2^{k+1}d$,

$$|\Omega(x-z)|\frac{|x-y|}{|x-z|^{n+1}}\le\|\Omega\|_\infty\frac{|x-y|}{(2^kd)^{n+1}}.$$

In addition, the homogeneity of degree 0 of Ω yields that

$$\begin{aligned}
|\Omega(y-z)-\Omega(x-z)|&=\left|\Omega\left(\frac{y-z}{|y-z|}\right)-\Omega\left(\frac{x-z}{|x-z|}\right)\right|\\
&=\left|\Omega\left(\frac{(x-z)-(x-y)}{|(x-z)-(x-y)|}\right)-\Omega\left(\frac{x-z}{|x-z|}\right)\right|.
\end{aligned}$$

Note $|x-z|\ge 2|x-y|$, so (2.1.11) implies that

$$|\Omega(y-z)-\Omega(x-z)|\le\omega_\infty\left(2\frac{|x-y|}{|x-z|}\right).$$

Since $\omega_\infty(\delta)$ is nondecreasing on δ, when $2^kd<|x-z|\le 2^{k+1}d$, we have that

$$\begin{aligned}
\frac{|\Omega(y-z)-\Omega(x-z)|}{|x-z|^n}&\le\frac{1}{(2^kd)^n}\omega_\infty\left(2\frac{|x-y|}{|x-z|}\right)\\
&\le\frac{1}{(2^kd)^n}\omega_\infty\left(\frac{d}{2^{k-1}d}\right)\\
&=\frac{1}{(2^kd)^n}\omega_\infty\left(\frac{1}{2^{k-1}}\right).
\end{aligned}$$

Thus

$$
\begin{aligned}
&\sum_{k=2}^{\infty} \int_{2^k d<|x-z|\le 2^{k+1}d} \left| \frac{\Omega(y-z)}{|y-z|^n} - \frac{\Omega(x-z)}{|x-z|^n} \right| |f(z)|dz \\
&\le C \sum_{k=2}^{\infty} \int_{2^k d<|x-z|\le 2^{k+1}d} |\Omega(x-z)| \frac{|x-y|}{|x-z|^{n+1}} |f(z)|dz \\
&\quad + C \sum_{k=2}^{\infty} \int_{2^k d<|x-z|\le 2^{k+1}d} \frac{|\Omega(y-z)-\Omega(x-z)|}{|x-z|^n} |f(z)|dz \\
&\le C \sum_{k=2}^{\infty} \|\Omega\|_\infty \frac{|x-y|}{(2^k d)^{n+1}} \int_{|x-z|\le 2^{k+1}d} |f(z)|dz \\
&\quad + C \sum_{k=2}^{\infty} \frac{1}{(2^k d)^n} \omega_\infty\left(\frac{1}{2^{k-1}}\right) \int_{|x-z|\le 2^{k+1}d} |f(z)|dz \\
&\le C' Mf(x) + CMf(x) \sum_{k=2}^{\infty} \omega_\infty\left(\frac{1}{2^{k-1}}\right).
\end{aligned}
$$

However,

$$
\begin{aligned}
\sum_{k=2}^{\infty} \omega_\infty\left(\frac{1}{2^{k-1}}\right) &= \frac{1}{\log 2} \sum_{k=2}^{\infty} \omega_\infty\left(\frac{1}{2^{k-1}}\right) \int_{\frac{1}{2^{k-1}}}^{\frac{1}{2^{k-2}}} \frac{d\delta}{\delta} \\
&\le \frac{1}{\log 2} \sum_{k=2}^{\infty} \int_{\frac{1}{2^{k-1}}}^{\frac{1}{2^{k-2}}} \omega_\infty(\delta) \frac{d\delta}{\delta} \\
&\le \frac{1}{\log 2} \int_0^1 \omega_\infty(\delta) \frac{d\delta}{\delta} < \infty.
\end{aligned}
$$

All estimates above imply that

$$
\frac{1}{|Q|} \int_Q |T_\Omega f_2(y) - T_\Omega f_2(x)|dy \le CMf(x) \le C(M(|f|^s)(x))^{\frac{1}{s}},
$$

which combining with (2.1.21) leads to (2.1.20). ∎

In order to formulate the next lemma, we first give definitions of dyadic cubes, the dyadic Hardy-Littlewood maximal function, and the dyadic sharp maximal function.

Let $Q_0 = [0,1)^n$ be the unit cube in $\mathbb{R}^n$, and $\mathcal{Q}_0$ is the total of cubes which are translated from Q_0 by one unit. That is, the length of the side of the cubes in $\mathcal{Q}_0$ are all 1 and they are nonoverlapping whose union is $\mathbb{R}^n$.

Similarly, for $k \in \mathbb{Z}$, let $Q_k = [0, 2^k)^n$, and $\mathcal{Q}_k$ are the total cubes which are translated from Q_k by 2^k units. That is, the length of the side of the cubes in $\mathcal{Q}_k$ are all 2^k and they are nonoverlapping whose union is $\mathbb{R}^n$. Thus we have the following properties.

(i) For any fixed $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, there is only one cube $Q_k \ni x$.

(ii) For any given two dyadic cubes, either their intersection is empty, or one lies in the other.

(iii) Any one cube in $\mathcal{Q}_k$ must contain 2^n cubes in $\mathcal{Q}_{k-1}$.

(iv) When $k < j$, any cube in $\mathcal{Q}_k$ must lie in a unique cube in $\mathcal{Q}_j$.

Lemma 2.1.3 *Suppose $\omega \in A_\infty$ and there exists $0 < p_0 < \infty$ such that $Mf \in L^{p_0}(\omega)$. Then*

$$\int_{\mathbb{R}^n} (Mf(x))^p \,\omega(x)dx \le C \int_{\mathbb{R}^n} \left(M^\sharp f(x)\right)^p \omega(x)dx \tag{2.1.23}$$

holds for $p_0 \le p < \infty$.

Proof. Since $M^\sharp(|f|)(x) \le 2M^\sharp f(x)$, without loss of generality, we may assume that $f \ge 0$. For any $\lambda > 0$, we use the Calderón-Zygmund decomposition. By the property (ii) of dyadic cubes, if there exists a sequence of dyadic cubes satisfying $Q_1 \subset Q_2 \subset \cdots$ and $f_{Q_k} > \lambda$, then $\{\omega(Q_k)\}$ is uniformly bounded about k. In fact, for any $x \in Q_k$, we have

$$t < \frac{1}{|Q_k|} \int_{Q_k} f(y)dy \le Mf(x).$$

Thus

$$\begin{aligned}\omega(Q_k) = \int_{Q_k} \omega(x)dx &\le \frac{1}{t^{p_0}} \int_{Q_k} (Mf(x))^{p_0} \omega(x)dx \\ &\le \frac{1}{t^{p_0}} \|Mf\|^{p_0}_{L^{p_0}(\omega)}.\end{aligned}$$

Since $\omega(x)dx$ is a double measure and $\omega \in A_\infty$, thus $\{|Q_k|\}$ is uniformly bounded. Therefore all the dyadic cubes Q satisfying $f_Q > t$ are definitely included in a dyadic cube, which is called the maximal dyadic cube. Suppose that $\{Q_j\}$ is the set of all maximal dyadic cubes, then any Q_j must satisfy

$$t < \frac{1}{|Q_j|} \int_{Q_j} f(y)dy \le 2^n t.$$

Since $\{Q_j\}$ depend on λ, without loss of generality we denote it by $\{Q_{\lambda,j}\}$. Note that when $\lambda < \mu$, for any $Q_{\mu,j}$, there must be a k such that $Q_{\mu,j} \subset Q_{\lambda,k}$.

Now for any $\lambda > 0$, let $Q_0 = Q_{2^{-n-1}\lambda, j_0}$ and $A > 0$. If

$$Q_0 \subset \left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\},$$

then

$$\sum_{\{j: Q_{\lambda,j} \subset Q_0\}} \omega(Q_{\lambda,j}) \le \omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right). \tag{2.1.24}$$

If $Q_0 \not\subseteq \left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}$, then

$$\frac{1}{|Q_0|}\int_{Q_0} |f(y) - f_{Q_0}| dy \le \frac{\lambda}{A}$$

and

$$f_{Q_0} = \frac{1}{|Q_0|}\int_{Q_0} f(t) dt \le 2^n 2^{-n-1}\lambda = \frac{\lambda}{2}.$$

Thus

$$\begin{aligned}
\sum_{\{j: Q_{\lambda,j} \subset Q_0\}} \left(\lambda - \frac{\lambda}{2}\right)|Q_{\lambda,j}| &= \sum_{\{j: Q_{\lambda,j} \subset Q_0\}} \int_{Q_{\lambda,j}} \left(\lambda - \frac{\lambda}{2}\right) dy \\
&\le \sum_{\{j: Q_{\lambda,j} \subset Q_0\}} \int_{Q_{\lambda,j}} |f(y) - f_{Q_0}| dy \\
&\le \int_{Q_0} |f(y) - f_{Q_0}| dy \\
&\le A^{-1}\lambda |Q_0|,
\end{aligned}$$

which gives

$$\sum_{\{j: Q_{\lambda,j} \subset Q_0\}} |Q_{\lambda,j}| \le 2A^{-1}|Q_0|.$$

On the other hand, since

$$\bigcup_{\{j: Q_{\lambda,j} \subset Q_0\}} Q_{\lambda,j} \subset Q_0$$

and $\omega \in A_\infty$, there exist $\delta > 0$ and $C > 0$, such that

$$\frac{\omega\left(\bigcup_{\{j:Q_{\lambda,j}\subset Q_0\}} Q_{\lambda,j}\right)}{\omega(Q_0)} \le C\left(\frac{\left|\sum_{\{j:Q_{\lambda,j}\subset Q_0\}} Q_{\lambda,j}\right|}{|Q_0|}\right)^{\delta}.$$

Thus we get

$$\sum_{\{j:Q_{\lambda,j}\subset Q_0\}} \omega(Q_{\lambda,j}) \le C\left(2A^{-1}\right)^{\delta}\omega(Q_0). \tag{2.1.25}$$

By (2.1.24) and (2.1.25) we have

$$\sum_{\{j:Q_{\lambda,j}\subset Q_0\}} \omega(Q_{\lambda,j}) \le \omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) + C\left(2A^{-1}\right)^{\delta}\omega(Q_0).$$

Now taking over all the Q_0, this inequality yields

$$\sum_j \omega(Q_{\lambda,j}) \le \omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) + C\left(2A^{-1}\right)^{\delta}\sum_k \omega\left(Q_{2^{-n-1}\lambda,k}\right). \tag{2.1.26}$$

Let $\alpha(\lambda) = \sum_j \omega\left(Q_{\lambda,j}\right)$ and $\beta(\lambda) = \omega(\{x : Mf(x) > \lambda\})$, then $\bigcup_j Q_{\lambda,j} \subset \{x : Mf(x) > \lambda\}$ and the disjoint of $\{Q_{\lambda,j}\}$ imply

$$\alpha(\lambda) \le \beta(\lambda). \tag{2.1.27}$$

Next we wish to obtain that

$$\beta(\lambda) \le \sum_j \omega\left(3Q_{4^{-n}\lambda,j}\right) \le C_1\alpha\left(\frac{\lambda}{C_2}\right). \tag{2.1.28}$$

Put $E_\lambda = \{x : Mf(x) > \lambda\}$, where $3Q$ is the cube whose side length is 3 times that of Q with the center at the same as that of Q. If we can show

$$E_\lambda \subset \bigcup_j 3Q_{4^{-n}\lambda,j}, \tag{2.1.29}$$

then the left inequality of (2.1.28) naturally holds. Therefore (2.1.28) holds, since the second one is clear. Now for any given $x \in E_\lambda$, R is a cube which contains x and satisfies

$$\frac{1}{|R|}\int_R f(y)dy > \lambda.$$

If we denote d as the side length of R, then there exists the unique integer k such that $2^{k-1} < d \le 2^k$. Thus in $\mathcal{Q}_k$ there are at most 2^n dyadic cubes that intersect with R, and there are at most one dyadic cube Q such that

$$\int_{R\cap Q} |f(y)|dy > \frac{\lambda|R|}{2^n}.$$

Since $|R| \le |Q| < 2^n|R|$, we have that

$$\int_{R\cap Q} |f(y)|dy > \frac{\lambda|R|}{2^n} > \frac{\lambda}{4^n}|Q|.$$

This is equivalent to

$$\frac{1}{|Q|}\int_Q |f(y)|dy > \frac{\lambda}{4^n}.$$

Consequently there exists j such that the maximal dyadic cube $Q_{4^{-n}\lambda,j} \supset Q$. Since $Q \cap R \ne \phi$ and $|R| \le |Q|$, then $R \subset 3Q \subset 3Q_{4^{-n}\lambda,j}$, which establishes (2.1.29). It follows from (2.1.26) that

$$\alpha(\lambda) \le \omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) + C\left(2A^{-1}\right)^\delta \alpha\left(2^{-n-1}\lambda\right). \tag{2.1.30}$$

For $N > 0$, by (2.1.27), applying the following equality

$$\int_0^\infty p_0\lambda^{p_0-1}\beta(\lambda)d\lambda = \int_{\mathbb{R}^n} [Mf(x)]^{p_0}\omega(x)dx,$$

we conclude that

$$\begin{aligned}
I_N &= \int_0^N p\lambda^{p-1}\alpha(\lambda)d\lambda \\
&\le \int_0^N p\lambda^{p-1}\beta(\lambda)d\lambda \\
&\le pp_0^{-1}N^{p-p_0}\int_0^N p_0\lambda^{p_0-1}\beta(\lambda)d\lambda \\
&\le pp_0^{-1}N^{p-p_0}\int_{\mathbb{R}^n} (Mf(x))^{p_0}\omega(x)dx \\
&< \infty.
\end{aligned}$$

On the other hand, (2.1.30) implies that

$$\begin{aligned}
I_N &\le \int_0^N p\lambda^{p-1}\omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) d\lambda \\
&\qquad + C\left(2A^{-1}\right)^\delta \int_0^N p\lambda^{p-1}\alpha(2^{-n-1}\lambda)d\lambda \\
&= \int_0^N p\lambda^{p-1}\omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) d\lambda \\
&\qquad + C2^{(n+1)p}\left(2A^{-1}\right)^\delta \int_0^{2^{-n-1}N} p\lambda^{p-1}\alpha(\lambda)d\lambda \\
&\le \int_0^N p\lambda^{p-1}\omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) d\lambda + C2^{(n+1)p}\left(2A^{-1}\right)^\delta I_N.
\end{aligned}$$

Now we take $A > 0$ such that $C2^{(n+1)p}(2A^{-1})^\delta = \frac{1}{2}$, then

$$I_N \le 2\int_0^N p\lambda^{p-1}\omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) d\lambda.$$

As N tends to ∞, we have

$$\int_0^\infty p\lambda^{p-1}\alpha(\lambda)d\lambda \le 2\int_0^\infty p\lambda^{p-1}\omega\left(\left\{x : M^\sharp f(x) > \frac{\lambda}{A}\right\}\right) d\lambda.$$

This together with (2.1.28) implies that

$$\begin{aligned}
\int_{\mathbb{R}^n}[Mf(x)]^p\omega(x)dx &= \int_0^\infty p\lambda^{p-1}\beta(\lambda)d\lambda \\
&\le C_1\int_0^\infty p\lambda^{p-1}\alpha\left(\frac{\lambda}{C_2}\right)d\lambda \\
&= C\int_0^\infty p\lambda^{p-1}\alpha(\lambda)d\lambda \\
&\le C\int_0^\infty p\lambda^{p-1}\omega(\{x : M^\sharp f(x) > \lambda\})d\lambda \\
&= C\int_{\mathbb{R}^n}\left(M^\sharp f(x)\right)^p\omega(x)dx.
\end{aligned}$$

∎

Theorem 2.1.6 *Suppose that Ω is a homogeneous bounded function of degree 0 and satisfies (2.1.8) and (2.1.9). Then there exists a constant C independent of f, for $1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$, such that*

$$\int_{\mathbb{R}^n}|T_\Omega f(x)|^p\omega(x)dx \le C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx. \tag{2.1.31}$$

Proof. Since $\omega \in A_p$, there exists $1 < s < p$ such that $\omega \in A_{\frac{p}{s}}$. First we will prove (2.1.31) under the condition that f is a bounded function with compact support. To the end, we will first illustrate $M(T_\Omega f)(x) \in L^p(\omega)$. It suffices to prove $T_\Omega f \in L^p(\omega)$. Without loss of generality we may assume that $\operatorname{supp}(f) \subset \{y : |y| < R\}$. If $|x| > 2R$, then

$$|x-y| \geq |x| - |y| > \frac{|x|}{2}.$$

It follows that

$$|T_\Omega f(x)| \leq \int_{|y|<R} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \leq \frac{C}{|x|^n} \|f\|_\infty. \tag{2.1.32}$$

Set

$$\begin{aligned}\int_{\mathbb{R}^n} |T_\Omega f(x)|^p \omega(x) dx &= \int_{|x|\leq 2R} |T_\Omega f(x)|^p \omega(x) dx + \int_{|x|>2R} |T_\Omega f(x)|^p \omega(x) dx \\ &:= I_1 + I_2.\end{aligned}$$

For some $\varepsilon > 0$, Hölder's inequality implies that

$$I_1 \leq \left(\int_{|x|\leq 2R} |T_\Omega f(x)|^{p(1+\varepsilon)/\varepsilon} dx \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_{|x|\leq 2R} \omega(x)^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}}.$$

Then the L^q−boundedness of T_Ω (to see Theorem 2.1.2) implies that the first part of right side of the above inequality is bounded. Since ω satisfies the inverse Hölder inequality, as ε is small enough, we deduce that the second part of right side of above inequality is also bounded. In order to illustrate that I_2 is bounded, we need the following property of A_p weight. If $\omega \in A_p$ $(1 \leq p < \infty)$, then for any cube $Q \subset \mathbb{R}^n$ and $a > 1$ we have

$$\omega(aQ) \leq C a^{np} \omega(Q), \tag{2.1.33}$$

where C only depends on ω but not on a. In fact, for any $f \in L^p(\omega)$ and $\lambda > 0$, Theorem 1.4.2 implies that

$$\omega(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$

Now we take $f \geq 0$ and cube Q such that

$$\int_Q f(y) dy > 0.$$

Thus for all $0 < \lambda < f_Q$, we have $Q \subset \{x : M(f\chi_Q)(x) > \lambda\}$, which implies that

$$\omega(Q) \leq \frac{C}{\lambda^p} \int_Q |f(x)|^p \omega(x)dx.$$

So

$$\omega(Q)\lambda^p \leq C \int_Q |f(x)|^p \omega(x)dx.$$

Now let $\lambda \to f_Q$, then

$$\omega(Q)\,(f_Q)^p \leq C \int_Q |f(x)|^p \omega(x)dx.$$

If we take $f = \chi_S$ with $S \subset Q$ being measurable set, then the above inequality becomes

$$\omega(Q) \left(\frac{|S|}{|Q|} \right)^p \leq C\omega(S).$$

Replacing Q by aQ and S by Q in the above formula will yield (2.1.33). Clearly if we replace cube Q by a ball in (2.1.33), the inequality still holds.

Now we turn our attention to the estimate of I_2. For $\omega \in A_p$ there exists $1 < q < p$ such that $\omega \in A_q$. From (2.1.32) and (2.1.33), it follows that

$$\begin{aligned}
I_2 &\leq C \sum_{k=1}^{\infty} \int_{2^k R < |x| \leq 2^{k+1}R} \frac{\omega(x)}{|x|^{np}} dx \\
&\leq C \sum_{k=1}^{\infty} \left(2^k R\right)^{-np} \omega\left(B(0, 2^{k+1}R)\right) \\
&\leq C \sum_{k=1}^{\infty} \left(2^k R\right)^{-np} C \left(2^{k+1}R\right)^{nq} \omega(B(0,1)) \\
&\leq C(n, R, \omega) < \infty.
\end{aligned}$$

We have proved that $T_\Omega f \in L^p(\omega)$, therefore $M(T_\Omega f) \in L^p(\omega)$. If f is a bounded function with compact support, then applying Lemma 2.1.2 and

Lemma 2.1.3 we have

$$\begin{aligned}\int_{\mathbb{R}^n} |T_\Omega f(x)|^p \omega(x)dx &\le \int_{\mathbb{R}^n} (M(T_\Omega f)(x))^p\, \omega(x)dx\\ &\le C\int_{\mathbb{R}^n} \left(M^\sharp(T_\Omega f)(x)\right)^p \omega(x)dx\\ &\le C\int_{\mathbb{R}^n} (M(|f|^s)(x))^{\frac{p}{s}}\, \omega(x)dx\\ &\le C\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx.\end{aligned}$$

Finally, since the set of bounded functions with compact support is dense in $L^p(\omega)$, T_Ω can be extended to $L^p(\omega)$ continuously. ∎

Theorem 2.1.7 *Suppose that Ω satisfies the conditions of Theorem 2.1.6 and $\omega \in A_1$. Then there exists a constant C, for any $\lambda > 0$ and $f \in L^1(\omega)$, such that*

$$\omega(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) \le \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|\omega(x)dx.$$

Proof. The proof is similar to that of truncated operator T_ε being of weak type (1,1) in Theorem 2.1.1. Using the Calderón-Zygmund decomposition of f and λ, we have $f = g + b$ and a series of nonoverlapping cubes $\{Q_k\}$. Clearly

$$\begin{aligned}\omega(\{x : |T_\Omega f(x)| > \lambda\}) &\le \omega\left(\left\{x : |T_\Omega g(x)| > \frac{\lambda}{2}\right\}\right) + \omega\left(\left\{x : |T_\Omega b(x)| > \frac{\lambda}{2}\right\}\right)\\ &:= I_1 + I_2.\end{aligned}$$

We will give the estimate of I_1 and I_2 respectively. Since $\omega \in A_1 \subset A_2$ and T_Ω is bounded on $L^2(\omega)$, we have

$$\begin{aligned}I_1 &\le \frac{4}{\lambda^2}\int_{\mathbb{R}^n} |T_\Omega g(x)|^2\omega(x)dx\\ &\le \frac{4C}{\lambda^2}\int_{\mathbb{R}^n} |g(x)|^2\omega(x)dx\\ &\le \frac{2^n 4C}{\lambda}\int_{\mathbb{R}^n} |g(x)|\omega(x)dx.\end{aligned}$$

Noting the definition of g in the Calderón-Zygmund decomposition, then

$$\begin{aligned}
I_1 &\leq \frac{C}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup Q_k}|f(x)|\omega(x)dx+\frac{C}{\lambda}\sum_k\int_{Q_k}\frac{1}{|Q_k|}\int_{Q_k}|f(y)|dy\omega(x)dx\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup Q_k}|f(x)|\omega(x)dx+\frac{1}{\lambda}\sum_k\int_{Q_k}|f(y)|\frac{\omega(Q_k)}{|Q_k|}dy\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup Q_k}|f(x)|\omega(x)dx+\frac{C}{\lambda}\sum_k\int_{Q_k}|f(y)|\omega(y)dy\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(x)|\omega(x)dx.
\end{aligned}$$

To estimate I_2, we will use some ideas from the proof of Lemma 2.1.2. For every Q_k, let B_k^* be the ball of which the center is the same as Q_k and the diameter is 16 times of that of Q_k. Then by (2.1.33), we obtain that

$$\begin{aligned}
\omega\left(\bigcup_k B_k^*\right) &\leq \sum_k \omega(B_k^*)\\
&\leq C\sum_k\omega(Q_k)\\
&\leq C\sum_k\frac{\omega(Q_k)}{|Q_k|}|Q_k|\\
&\leq C\sum_k\frac{\omega(Q_k)}{|Q_k|}\frac{1}{\lambda}\int_{Q_k}|f(y)|dy\\
&\leq \frac{C}{\lambda}\sum_k\int_{Q_k}|f(y)|\omega(y)dy\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(y)|\omega(y)dy.
\end{aligned}$$

Finally, we will give the estimate of

$$\omega\left(\left\{x\in\mathbb{R}^n\setminus\bigcup_k B_k^*:|T_\Omega b(x)|>\frac{\lambda}{2}\right\}\right).$$

Denote the center of Q_k by c_k. Since $\int_{Q_k}b_k(x)dx=0$ for every k, we have

that

$$\begin{aligned}
&\omega\left(\left\{x\in\mathbb{R}^n\backslash\bigcup_k B_k^*:|T_\Omega b(x)|>\frac{\lambda}{2}\right\}\right)\\
&\le\frac{C}{\lambda}\sum_k\int_{\mathbb{R}^n\backslash B_k^*}|T_\Omega b_k(x)|\omega(x)dx\\
&=\frac{C}{\lambda}\sum_k\int_{\mathbb{R}^n\backslash B_k^*}\left|\int_{Q_k}\left[\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x-c_k)}{|x-c_k|^n}\right]b_k(y)dy\right|\omega(x)dx\\
&\le\frac{C}{\lambda}\sum_k\int_{Q_k}|b_k(y)|\left(\int_{\mathbb{R}^n\backslash B_k^*}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x-c_k)}{|x-c_k|^n}\right|\omega(x)dx\right)dy.
\end{aligned}$$

Using the method in the proof of (2.1.22) and the property of A_1. it is not hard to get that

$$\int_{\mathbb{R}^n\backslash B_k^*}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x-c_k)}{|x-c_k|^n}\right|\omega(x)dx\le CM\omega(y)\le C\omega(y)\tag{2.1.34}$$

for any $y\in Q_k$. Thus by (2.1.34), we obtain that

$$\begin{aligned}
\omega\left(\left\{x\in\mathbb{R}^n\backslash\bigcup_k B_k^*:|T_\Omega b(x)|>\frac{\lambda}{2}\right\}\right)&\le\frac{C}{\lambda}\sum_k\int_{Q_k}|b_k(y)|\omega(y)dy\\
&\le\frac{C}{\lambda}\int_{\mathbb{R}^n}(|f(y)|+|g(y)|)\omega(y)dy\\
&\le\frac{C}{\lambda}\int_{\mathbb{R}^n}|f(y)|\omega(y)dy.
\end{aligned}$$

Using the Cotlar inequality (Lemma 2.1.1), the weighed norm inequality of the Hardy-Littlewood maximal operator M (Theorem 1.4.2 and Theorem 1.4.3) as well as the weighed norm inequality of the Calderón-Zygmund singular integral operator T_Ω, we can immediately obtain the weighed norm inequality of the maximal singular integral operator. ■

Theorem 2.1.8 *Suppose that Ω is a homogeneous bounded function of degree 0 and satisfies (2.1.8) and (2.1.9). Then T_Ω^* is bounded on $L^p(\omega)$ for $1<p<\infty$ and $\omega\in A_p$, and T_Ω^* is bounded from $L^1(\omega)$ to $L^{1,\infty}(\omega)$ with $\omega\in A_1$.*

Corollary 2.1.3 *Let $1\le p<\infty$ and $\omega\in A_p(\mathbb{R}^n)$. Then*

(i) H and H^* are both bounded from $L^p(\omega)$ to $L^p(\omega)$ for $1<p<\infty$;
(ii) H and H^* are both bounded from $L^1(\omega)$ to $L^{1,\infty}(\omega)$.

Corollary 2.1.4 *Let $1 \le p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$, R_j and R_j^* be the Riesz transform and the maximal Riesz transform respectively $(j = 1, 2, \cdots)$. Then (i) R_j and R_j^* are both bounded operator from $L^p(\omega)$ to $L^p(\omega)$ for $1 < p < \infty$; (ii) R_j and R_j^* are both bounded operators from $L^1(\omega)$ to $L^{1,\infty}(\omega)$.*

Next we are going to investigate the vector-valued singular integral operator and its applications. Suppose that B is a separable Banach space and B^* is the dual space of B. The mapping $f : \mathbb{R}^n \to B$ is called a B-value function. A B-value function f is called measurable, if for any $b' \in B^*$, the mapping $x \to \langle f(x), b' \rangle$ is measurable. If a B-value function f is measurable, then the function $x \to \|f(x)\|_B$ is also measurable. For $1 \le p \le \infty$, define

$$L^p(B) = \left\{ f : \|f\|_{L^p(B)} = \left(\int_{\mathbb{R}^n} \|f(x)\|_B^p dx \right)^{\frac{1}{p}} < \infty \right\}, 1 \le p < \infty,$$

$$L^\infty(B) = \left\{ f : \|f\|_{L^\infty(B)} = \sup \{ \|f\|_B : x \in \mathbb{R}^n \} < \infty \right\}.$$

$L^p(B)(1 \le p \le \infty)$ are all Banach spaces.

Similarly, weak $L^p(B)$ $(1 \le p < \infty)$ spaces are defined by

$$L_B^{p,\infty} = \left\{ f : \|f\|_{L^{p,\infty}(B)} = \sup_{t>0} t \left| \{ x \in \mathbb{R}^n : \|f(x)\|_B > t \} \right|^{\frac{1}{p}} < \infty \right\}.$$

We now formulate a dense subspace of $L^p(B)$. If $f \in L^p(\mathbb{R}^n)$ is a real-valued function and $b \in B$, then the function $(f \cdot b) \in L^p(B)$, where $(f \cdot b)(x) := f(x)b$. The set of all the finite linear combination of all the vector-valued functions of this form is a dense subspace of $L^p(B)$, and we denote it by $L^p \bigotimes B$.

For

$$F = \sum_j f_j b_j \in L^1 \bigotimes B,$$

its integral is defined by

$$\int_{\mathbb{R}^n} F(x)dx = \sum_j \left(\int_{\mathbb{R}^n} f_j(x)dx \right) b_j \in B.$$

Thus since $L^1 \bigotimes B$ is dense in $L^1(B)$, the above definition can be continuously extended to $L^1(B)$. And for $F \in L^1(B)$, its integral on $\mathbb{R}^n$ is the unique member in B satisfying the following equation

$$\left\langle \int_{\mathbb{R}^n} F(x)dx, b' \right\rangle = \int_{\mathbb{R}^n} \langle F(x), b' \rangle dx$$

for every $b' \in B^*$. Suppose that $F(x) \in L^p(B), G \in L^{p'}(B^*)$, then $\langle F(x), G(x)\rangle$ is an integrable function on $\mathbb{R}^n$ and

$$\|G\|_{L^{p'}(B^*)} = \sup\left\{\left|\int_{\mathbb{R}^n}\langle F(x), G(x)\rangle dx\right| : \|F\|_{L^p(B)} \le 1\right\}.$$

Especially, when $1 \le p < \infty$ and B is a reflexible space, $L^{p'}(B^*) = (L^p(B))^*$.

Suppose that both A and B are two Banach spaces. Denote $\mathscr{L}(A, B)$ to be the set of all bounded linear operators from A to B. Suppose that K is a $\mathscr{L}(A, B)$-valued function defined on $\mathbb{R}^n\backslash\{0\}$. Let $f \in L^\infty(A)$ be A-valued function and be compactly supported. Then the operator T associated to K is defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy, \qquad x \notin \text{supp}f.$$

This operator T is called vector-valued singular integral operator.

To obtain the L^p boundedness of vector-valued singular integral operator, we first introduce a lemma.

Lemma 2.1.4 *Suppose that T is a linear operator on $L^\infty(A)$ defined above with compact support.*

(i) There exists $1 < p_0 < \infty$ such that

$$|E_\lambda(Tf)| \le \frac{C}{\lambda^{p_0}}\|f\|^{p_0}_{L^{p_0}(A)},$$

where $\lambda > 0$ and $E_\lambda(Tf) = \{x \in \mathbb{R}^n : \|Tf(x)\|_B > \lambda\}$;

(ii) Let $U = U(x_0, r)$ be the ball with the center at x_0 and radius r and $2U$ be the twice expansion of U at the same center. For all functions f in $L^1(A)$ satisfying supp$(f) \subset U$ and $\int_{\mathbb{R}^n} f(x)dx = 0$, we have

$$\int_{\mathbb{R}^n\backslash 2U} \|Tf(x)\|_B dx \le C\|f\|_{L^1(A)}. \tag{2.1.35}$$

If both (i) and (ii) hold, then $\|Tf\|_{L^p(B)} \le C\|f\|_{L^p(A)}$ holds for $1 < p < p_0$, where C is independent of f.

Proof. We will first prove that, for any $\lambda > 0$,

$$|E_\lambda(Tf)| \leq \frac{C}{\lambda}\|f\|_{L^1(A)}. \tag{2.1.36}$$

For $\lambda > 0$, using the Calderón-Zygmund decomposition to real-valued function $\|f(x)\|_A$, we get a seguence $\{Q_j\}$ with nonoverlapping cubes. Now we define A-value functions g and h such that $f = g + h$ with

$$g(x) = \begin{cases} f(x), & x \notin \bigcup_j Q_j; \\ \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f(x)dx & x \in Q_j, \end{cases}$$

$$\int_{Q_j} h(x)dx = 0, \qquad \text{for every } j.$$

Thus it follows that

$$\|g\|_A \leq 2^n\lambda,$$

$$\|g\|_{L^1(A)} \leq \|f\|_{L^1(A)}$$

and

$$\|h\|_{L^1(A)} \leq 2\|f\|_{L^1(A)}.$$

Thus from the condition (i) it follows that

$$\begin{aligned} \left|E_{\frac{\lambda}{2}}(Tg)\right| &\leq \frac{2^{p_0}C}{\lambda^{p_0}}\|g\|^{p_0}_{L^{p_0}(A)} \\ &\leq \frac{2^{p_0}C}{\lambda^{p_0}}\int_{\mathbb{R}^n}(2^n\lambda)^{p_0-1}\|g(x)\|_A dx \\ &\leq \frac{C}{\lambda}2^{n(p_0-1)+p_0}\|g\|_{L^1(A)} \\ &\leq \frac{C2^{n(p_0-1)+p_0}}{\lambda}\|f\|_{L^1(A)}. \end{aligned}$$

Let S_j be a ball with the same center and radius as Q_j, $S_j^* = 2S_j$ and $D_1 = \bigcup_j S_j^*$. Put $h_j = h\chi_{Q_j}$. Then $\displaystyle\sum_{j=1}^N h_j$ converges to h in $L^{p_0}(A)$. It follows from condition (i) that

$$T\left(\sum_{j=1}^N h_j\right) \to Th, \text{ as } N \to \infty$$

in the sense of measure. Thus

$$\|Th\|_B \le \sum_{j=1}^{\infty} \|Th_j\|_B.$$

Consequently, it follows from the condition (ii) that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus D_1} \|Th\|_B dx &\le \sum_{j=1}^{\infty} \int_{\mathbb{R}^n\setminus D_1} \|Th_j\|_B dx \\ &\le \sum_{j=1}^{\infty} \int_{\mathbb{R}^n\setminus S_j^*} \|Th_j\|_B dx \\ &\le C\sum_{j=1}^{\infty} \|h_j\|_{L^1(A)} \\ &\le C'\|f\|_{L^1(A)}.\end{aligned}$$

On the other hand, we have

$$|D_1| \le \sum_j |S_j^*| \le C2^n \sum_j |Q_j| \le C_n \sum_j \frac{1}{\lambda}\int_{Q_j} \|f\|_A dx \le \frac{C_n}{\lambda}\|f\|_{L^1(A)}.$$

Thus

$$\left|E_{\frac{\lambda}{2}}(Th)\right| \le |D_1| + \frac{2}{\lambda}\int_{\mathbb{R}^n\setminus D_1} \|Th\|_B dx \le \frac{C}{\lambda}\|f\|_{L^1(A)}.$$

From above it follows (2.1.36). By (2.1.36) and the condition (i) as well as using the Marcinkiewicz interpolation theorem, it follows that

$$\|Tf\|_{L^p(B)} \le C\|f\|_{L^p(A)}$$

for $1 < p < p_0$. ■

Theorem 2.1.9 *Let A and B be two reflexive Banach spaces. Suppose that there exists a p_0, $1 < p_0 < \infty$, such that T is a bounded operator from $L^{p_0}(A)$ to $L^{p_0}(B)$ and $K(x)$ satisfies*

$$\int_{|x|\ge 2|y|} \|K(x-y) - K(x)\|_{\mathscr{L}(A,B)} dx \le C, \tag{2.1.37}$$

Then T is bounded from $L^p(A)$ to $L^p(B)$ with $1 < p < \infty$ and T is of weak type (1,1), i.e., there is $C > 0$ such that, for any $\lambda > 0$ and $f \in L^1(A)$,

$$|E_\lambda(Tf)| = |\{x \in \mathbb{R}^n : \|Tf(x)\|_B > \lambda\}| \le \frac{C}{\lambda}\|f\|_{L^1(A)}.$$

Proof. By the given condition we can easily deduce that T is of weak (p_0, p_0) type, i.e., for any $\lambda > 0$,

$$|E_\lambda(Tf)| \le \frac{C}{\lambda^{p_0}} \|f\|^{p_0}_{L^{p_0}(A)}.$$

Next we will show that T satisfies the condition (ii) in Lemma 2.1.4 by (2.1.37). Actually, suppose that a function f satisfies that $\operatorname{supp}(f) \subset U(x_0, r)$ and its mean value is zero. From (2.1.37) it follows that

$$\begin{aligned}\int_{\mathbb{R}^n\backslash 2U} \|Tf(x)\|_B dx &= \int_{\mathbb{R}^n\backslash 2U} \left\| \int_{\mathbb{R}^n} K(x-y)f(y)dy \right\|_B dx \\ &= \int_{|x|>2r} \left\| \int_{\mathbb{R}^n} K(x-y)f(y-x_0)dy \right\|_B dx \\ &= \int_{|x|>2r} \left\| \int_{\mathbb{R}^n} [K(x-y)-K(x)]f(y-x_0)dy \right\|_B dx \\ &\le \int_{|y|\le r} \|f(y-x_0)\|_A \int_{|x|>2r} \|K(x-y)-K(x)\|_{\mathscr{L}(A,B)} \\ &\quad \times dxdy \\ &\le C\|f\|_{L^1(A)}.\end{aligned}$$

Thus applying Lemma 2.1.4 we have

$$\|Tf\|_{L^p(B)} \le C\|f\|_{L^p(A)} \tag{2.1.38}$$

for $1 < p < p_0$.

Now let K^* be the conjugate operator of K. K^* is a function from $\mathbb{R}^n \to \mathscr{L}(B^*, A^*)$. For any function g in $L^\infty(B)$ with compact support, define linear operator

$$\overline{T}g(x) = \int_{\mathbb{R}^n} K^*(y-x)g(y)dy,$$

then $\overline{T}$ is the conjugate operator of T. It is easy to see that, for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|\overline{T}g\|_{L^{p'}(A^*)} \le C\|g\|_{L^{p'}(B^*)} \tag{2.1.39}$$

if and only if

$$\|Tf\|_{L^p(B)} \le C\|f\|_{L^p(A)}.$$

It follows that $\overline{T}$ is a bounded operator from $L^{p_0'}(B^*)$ to $L^{p_0'}(A^*)$. Clearly (2.1.37) implies that K^* satisfies

$$\int_{|x|\ge 2|y|} \|K^*(y-x) - K^*(x)\|_{\mathscr{L}(B^*,A^*)} dx \le C.$$

Thus, for $1 < q < p_0'$, we have

$$\|\overline{T}g\|_{L^q(A^*)} \le C\|g\|_{L^q(B^*)}.$$

This, together with (2.1.39), yields that

$$\|Tf\|_{L^p(B)} \le C\|f\|_{L^p(A)}, \qquad p_0 < p < \infty.$$

Thus we have proved that T is a bounded linear operator from $L^p(A)$ to $L^p(B)$ $(1 < p < \infty)$. Thus Lemma 2.1.4 implies that T is of weak type (1,1). ■

Remark 2.1.6 *For simplicity of the proof, we add a condition on reflexivite in Theorem 2.1.9. However, it should be pointed out that the condition on the reflexivite of A and B in Theorem 2.1.9 can be removed (see [RuRT]).*

Next we will give some application of Theorem 2.1.9.

Theorem 2.1.10 *Suppose that K satisfies the Hörmander condition (2.1.3) and T is a convolution operator generated by K. If T is of type (2, 2), then for any $r, p, 1 < p, r < \infty$, we have*

$$\left\| \left(\sum_j |Tf_j|^r \right)^{\frac{1}{r}} \right\|_p \le C_{p,r} \left\| \left(\sum_j |f_j|^r \right)^{\frac{1}{r}} \right\|_p .$$

When $p = 1$, we have

$$\left| \left\{ x \in \mathbb{R}^n : \left(\sum_j |Tf_j(x)|^r \right)^{\frac{1}{r}} > \lambda \right\} \right| \le \frac{C_r}{\lambda} \left\| \left(\sum_j |f_j|^r \right)^{\frac{1}{r}} \right\|_1 .$$

Proof. It is clear that T is operator of type (p,p) $(1 < p < \infty)$ under the given conditions. Take $A = B = l^r$ and let $\widetilde{f} = (f_1, f_2, \cdots) \in L^p(l^r)$. Then $\widetilde{T}$ is bounded from $L^r(l^r)$ to $L^r(l^r)$, where $\widetilde{T}\widetilde{f} = (Tf_1, Tf_2, \cdots)$. The kernel corresponding to $\widetilde{T}$ is $K \bigotimes I$, where I is the identity operator on l^r. Thus,

$$\int_{|x|>2|y|} \|(K(x-y) - K(x))I\|_{\mathscr{L}(l^r,l^r)}\, dx = \int_{|x|\ge 2|y|} |K(x-y) - K(x)| dx \le C.$$

Thus Theorem 2.1.10 holds. ■

Another application of Theorem 2.1.9 is to show the boundedness of the Littlewood-Paley operator.

Theorem 2.1.11 *Suppose that $\varphi(x) \in L^1(\mathbb{R}^n)$ satisfies the following conditions:*

(i) $\displaystyle\int_{\mathbb{R}^n} \varphi(x)dx = 0$;

(ii) $|\varphi(x)| \leq C(1+|x|)^{-n-\alpha}, \qquad x \in \mathbb{R}^n$;

(iii) $\displaystyle\int_{\mathbb{R}^n} |\varphi(x+h) - \varphi(x)|dx \leq C|h|^\alpha, \qquad h \in \mathbb{R}^n$,

for some $C > 0$ and $\alpha > 0$. Then the Littlewood-Paley operators

$$Gf(x) = \left(\sum_{j=-\infty}^{\infty} |\varphi_{2^j} * f(x)|^2\right)^{\frac{1}{2}}$$

and

$$\triangle f(x) = \left(\int_0^\infty |\varphi_t * f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

are of type (p,p) $(1 < p < \infty)$ and of weak type (1, 1), where

$$\varphi_{2^j}(x) = \frac{1}{2^{jn}}\varphi\left(\frac{x}{2^j}\right)$$

and

$$\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right).$$

Proof. For the operator G, let $A = \mathbb{C}, B = l^2$ and $K(x) = \{\varphi_{2^j}(x)\}_{j=-\infty}^{\infty}$, then by the Plancherel Theorem and Theorem 2.1.9, if we can prove , there exists $C > 0$ such that

$$\sum_j |\widehat{\varphi_{2^j}}(\xi)|^2 \leq C, \qquad \xi \in \mathbb{R}^n \tag{2.1.40}$$

and

$$\int_{|x|\geq 2|y|} \left(\sum_j |\varphi_{2^j}(x-y) - \varphi_{2^j}(x)|^2\right)^{\frac{1}{2}} dx \leq C, \qquad y \in \mathbb{R}^n, \tag{2.1.41}$$

then the operator G is of type (p,p) and of weak type (1,1) $(1 < p < \infty)$. Similarly, for the operator $\triangle$, take $B = L^2\left(\mathbb{R}_+, \frac{dt}{t}\right)$, then if we can prove that there exists $C > 0$ such that

$$\int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} \leq C, \qquad \xi \in \mathbb{R}^n \tag{2.1.42}$$

and

$$\int_{|x|\geq 2|y|}\left(\int_0^\infty |\varphi_t(x-y)-\varphi_t(x)|^2\frac{dt}{t}\right)^{\frac{1}{2}}dx\leq C,\qquad y\in\mathbb{R}^n,\quad (2.1.43)$$

then $\triangle$ is operator of type (p,p) $(1<p<\infty)$ and of weak type (1,1).

Since validations of the former and the latter are essentially the same, it suffices to prove (2.1.42) and (2.1.43). When $0<|\xi|<1$, from the vanishing condition (i) of φ, it follows that

$$\begin{aligned}|\widehat{\varphi}(\xi)| &= \left|\int_{\mathbb{R}^n}\varphi(x)\left(e^{-2\pi i x\cdot\xi}-1\right)dx\right|\\ &\leq \int_{|x|\leq|\xi|^{-\frac{1}{2}}}|\varphi(x)|\left|e^{-2\pi i x\cdot\xi}-1\right|dx + C\int_{|x|>|\xi|^{-\frac{1}{2}}}|\varphi(x)|dx\\ &\leq 2\pi\|\varphi\|_1|\xi|^{\frac{1}{2}}+C\int_{|\xi|^{-\frac{1}{2}}}^\infty\frac{dt}{t^{1+\alpha}}\\ &\leq C|\xi|^\beta,\end{aligned}$$

where $\beta=\inf\left\{\frac{1}{2},\frac{\alpha}{2}\right\}$.

When $|\xi|\geq 1$, note that

$$[\varphi(\cdot+h)-\varphi(\cdot)]\hat{}(\xi)=\widehat{\varphi}(\xi)\left(e^{2\pi i h\cdot\xi}-1\right).$$

Taking $h=\frac{\xi}{2|\xi|^2}$, from the above equality and the condition (iii) it follows that

$$2\left|\widehat{\varphi}(\xi)\right|\leq\int_{\mathbb{R}^n}|\varphi(x+h)-\varphi(x)|dx\leq C|h|^\alpha\leq C|\xi|^{-\alpha}.$$

For any $\xi\neq 0$, denote $\xi'=\frac{\xi}{|\xi|}$. Then

$$\int_0^\infty|\widehat{\varphi}(t\xi)|^2\frac{dt}{t}=\int_0^\infty\left|\widehat{\varphi}(t|\xi|\xi')\right|^2\frac{dt}{t}=\int_0^\infty\left|\widehat{\varphi}(t\xi')\right|^2\frac{dt}{t}.$$

The above estimates implies that

$$\int_0^\infty\left|\widehat{\varphi}(t\xi')\right|^2\frac{dt}{t}\leq\int_0^1 Ct^{2\beta-1}dt+\int_1^\infty Ct^{-1-\alpha}dt\leq C_1<\infty.$$

Next we will validate (2.1.43). It is easy to see that the left side of (2.1.43) is dilation invariant with respect to y. Thus we merely need to prove (2.1.43) in the case $|y| = 1$. The Schwarz inequality implies that

$$\begin{aligned}
&\int_{|x|>2} \left(\int_0^\infty |\varphi_t(x-y) - \varphi_t(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx \\
&\le \left(\int_{|x|>2} |x|^{-n-\frac{\alpha}{2}} dx \right)^{\frac{1}{2}} \left(\int_{|x|>2} \int_0^\infty |\varphi_t(x-y) - \varphi_t(x)|^2 |x|^{n+\frac{\alpha}{2}} \frac{dt}{t} dx \right)^{\frac{1}{2}} \\
&\le C_\alpha \left[\int_0^\infty t^{\frac{\alpha}{2}} \left(\int_{|x|>\frac{2}{t}} \left| \varphi\left(x - \frac{y}{t}\right) - \varphi(x) \right|^2 |x|^{n+\frac{\alpha}{2}} dx \right) \frac{dt}{t} \right]^{\frac{1}{2}}.
\end{aligned}$$

Now let

$$I(t) = \int_{|x|>\frac{2}{t}} \left| \varphi\left(x - \frac{y}{t}\right) - \varphi(x) \right|^2 |x|^{n+\frac{\alpha}{2}} dx.$$

If $|x| > \frac{2}{t}$ and $|y| = 1$, then we have

$$\left| x - \frac{y}{t} \right| \ge |x| - \frac{1}{t} \ge \frac{|x|}{2}.$$

When $0 < t < 1$, the condition (ii) yields

$$I(t) \le C \int_{|x|>\frac{2}{t}} \frac{dx}{|x|^{n+\frac{3\alpha}{2}}} \le C t^{\frac{3\alpha}{2}}.$$

On the other hand, take $0 < \varepsilon < \frac{\alpha}{2}$, by (ii) we have

$$\left| \phi\left(x - \frac{y}{t}\right) - \phi(x) \right| \le \frac{C}{(1+|x|)^{n+\alpha}} \le \frac{C}{(1+|x|)^{n+\varepsilon}} \le \frac{C}{|x|^{n+\varepsilon}}.$$

When $t \ge 1$, the condition (ii) gives, noting that $|x| > \frac{2}{t}$,

$$\left| \varphi\left(x - \frac{y}{t}\right) - \varphi(x) \right| \cdot |x|^{n+\frac{\alpha}{2}} \le C|x|^{\frac{\alpha}{2}-\varepsilon} \le C t^{\varepsilon-\frac{\alpha}{2}}.$$

Therefore applying the condition (iii) we have

$$I(t) \le C t^{\varepsilon-\frac{\alpha}{2}} \int_{|x|>\frac{2}{t}} \left| \varphi\left(x - \frac{y}{t}\right) - \varphi(x) \right| dx \le C t^{\varepsilon-\frac{3\alpha}{2}}.$$

Consequently

$$\int_0^\infty t^{\frac{\alpha}{2}} I(t) \frac{dt}{t} \le \int_0^1 C t^{\frac{\alpha}{2}+\frac{3\alpha}{2}-1} dt + \int_1^\infty C t^{\frac{\alpha}{2}-\frac{3\alpha}{2}+\varepsilon-1} dt < \infty,$$

which proves (2.1.43). ■

Remark 2.1.7 *(1) Clearly, if $\varphi \in \mathscr{S}(\mathbb{R}^n)$, then φ satisfies (ii) and (iii).*

(2) The conditions (ii) and (iii) can be weakened. Later we will see that there exists a class of functions in $L^1(\mathbb{R}^n)$, which only satisfies the vanishing condition (i) and weaker conditions on size, such that the Littlewood-Paley operators G and $\triangle$ are both of type $(p,p)(1 < p < \infty)$ and weak type (1,1).

The next consequence is a corollary of Theorem 2.1.9 and Theorem 2.1.11, and it is very useful in our following discussion.

Corollary 2.1.5 *Suppose that φ satisfies the conditions of Theorem 2.1.11. If $\overrightarrow{f} = (\cdots, f_{-1}, f_0, f_1, \cdots) \in L^p(l^2)(\mathbb{R}^n)$ for $1 < p < \infty$, then we have that*

$$\left\| \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\varphi_{2^k} * f_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{j\in\mathbb{Z}} |f_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_p \tag{2.1.44}$$

and

$$\left\| \left(\int_0^\infty \sum_{j\in\mathbb{Z}} |\varphi_t * f_j(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{j\in\mathbb{Z}} |f_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_p . \tag{2.1.45}$$

Proof. Let us now give the proof of (2.1.44). Let $A = l^2, B = l^2(l^2)$. we assert

$$\{a_{jk}\}_{j,k=-\infty}^{\infty} \in l^2(l^2),$$

provided that

$$\left\{ \left(\sum_{k\in\mathbb{Z}} |a_{jk}|^2 \right)^{\frac{1}{2}} \right\}_{j\in\mathbb{Z}} \in l^2.$$

Now for $\overrightarrow{f} \in L^2(l^2)(\mathbb{R}^n)$, define an operator T as follows:

$$T\overrightarrow{f}(x) = \left\{ \left(\sum_{k\in\mathbb{Z}} |\varphi_{2^k} * f_j(x)|^2 \right)^{\frac{1}{2}} \right\}_{j\in\mathbb{Z}} .$$

Then when $p=2$, Theorem 2.1.11 shows that

$$\begin{aligned}\left\|T\overrightarrow{f}\right\|^2_{L^2(B)} &= \int_{\mathbb{R}^n}\sum_j\sum_k |\varphi_{2^k} * f_j(x)|^2\,dx\\ &\le C\sum_j\int_{\mathbb{R}^n}|f_j(x)|^2dx\\ &= C\left\|\overrightarrow{f}\right\|^2_{L^2(A)}.\end{aligned}$$

Let $\widetilde{K}(x)$ be the kernel of T. Then $\widetilde{K}(x):\mathbb{R}^n\to\mathscr{L}(l^2,l^2(l^2))$. On the other hand, let $K(x)=\{\varphi_{2^k}(x)\}_{k\in\mathbb{Z}}$. Then for any $x\in\mathbb{R}^n$, we have $\left\|\widetilde{K}(x)\right\|_{\mathscr{L}(A,B)}\le\|K(x)\|_{\mathscr{L}(\mathbb{C},A)}$. Thus, we have that

$$\begin{aligned}&\int_{|x|\ge 2|y|}\left\|\widetilde{K}(x-y)-\widetilde{K}(x)\right\|_{\mathscr{L}(A,B)}dx\\ &\le\int_{|x|\ge 2|y|}\|K(x-y)-K(x)\|_{\mathscr{L}(\mathbb{C},A)}dx\\ &=\int_{|x|\ge 2|y|}\left(\sum_k|\varphi_{2^k}(x-y)-\varphi_{2^k}(x)|^2\right)^{\frac{1}{2}}dx\\ &\le C,\end{aligned}$$

for every $y\in\mathbb{R}^n$. The last inequality is derived from Theorem 2.1.11. Theorem 2.1.9 tells us that T is bounded operator from $L^p(A)$ to $L^p(B)$, i.e. (2.1.44) holds. ■

Remark 2.1.8 *(i) The conclusion of Theorem 2.1.9 shows that the corresponding inequality of weak type still holds under the condition of Corollary 2.1.5.*

(ii) The conclusion of Corollary 2.1.5 is still valid for $\varphi\in\mathscr{S}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n}\varphi(x)dx=0.$$

2.2 Singular integral operators with homogeneous kernels

In the first section of this chapter, we have studied the L^p boundedness $(1<p<\infty)$ and weak $(1,1)$ boundedness of the Calderón-Zygmund singular integral operators. However, it is noticed that under the condition on the kernel K, we require that K satisfies the size-condition (2.1.1) and the smooth condition (2.1.3) (i.e., Hörmander condition) besides the vanishing condition (2.1.2). As to singular integral operators with homogeneous kernel (for its definition, see (2.1.12)), besides vanishing condition (2.1.8), we still require that Ω satisfies L^∞-Dini condition (2.1.9). Then a natural question is raised up: can we weaken the condition of Ω but still ensure the L^p boundedness ($1 < p < \infty$) and weak $(1,1)$ boundedness of the corresponding singular integral operator T_Ω? In this section, we will discuss this problem.

Note that the operator T_Ω commutes with dilation δ_ε ($\varepsilon > 0$), so Ω should satisfy the homogeneous condition of degree 0, that is

$$\Omega(\lambda x) = \Omega(x) \tag{2.2.1}$$

holds for any $\lambda > 0$ and every $x \in \mathbb{R}^n$. In addition, from the discussion before Theorem 2.1.2, we know that $\Omega(x')$ must satisfy the vanishing condition on $\mathbb{S}^{n-1}$, i.e.,

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{2.2.2}$$

Thus we can only weaken the size-condition and smooth condition on Ω.

Let us now pay attention to investigating the L^p boundedness and weak type (1,1). Since $L^\infty(\mathbb{S}^{n-1}) \subsetneqq L^q(\mathbb{S}^{n-1})$ $(1 \le q < \infty)$, we first consider L^q-Dini condition instead of L^∞-Dini condition. Now we give the following definition.

Definition 2.2.1 (L^q-Dini condition) *We say a function $\Omega(x')$ on $\mathbb{S}^{n-1}$ satisfies L^q-Dini condition if*

(i) $\Omega \in L^q(\mathbb{S}^{n-1})(1 \le q < \infty)$,

(ii) $\displaystyle\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty$,

where $\omega_q(\delta)$ is called integral continuous modulus of Ω of degree q. Its definition is, for $\delta > 0$,

$$\omega_q(\delta) = \sup_{\|\rho\| < \delta} \left(\int_{\mathbb{S}^{n-1}} \left| \Omega(\rho x') - \Omega(x') \right|^q d\sigma(x') \right)^{1/q},$$

where ρ is a rotation in $\mathbb{R}^n$. $\|\rho\| = \sup \left\{ |\rho x' - x'| : x' \in S^{n-1} \right\}.$

Remark 2.2.1

1. If $\Omega(x')$ satisfies Lipschitz condition on S^{n-1}, that is, there exists a constant $M > 0$, such that $\forall x', y' \in S^{n-1}, |\Omega(x') - \Omega(y')| \le M|x' - y'|$, then Ω must satisfy L^q-Dini condition $(1 \le q \le \infty)$.

2. If $1 \le r < q \le \infty$, then Ω satisfies L^r-Dini condition whenever it satisfies L^q-Dini condition.

Suppose Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$. Put $K(x) = \Omega(x)|x|^{-n}$, $x \in \mathbb{R}^n$. The following theorem gives some important relations.

Theorem 2.2.1 *(a) If Ω satisfies L^1-Dini condition, then $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1})$ and K satisfies Hörmander condition, that is*

$$\int_{|x|\geqslant 2|y|} |K(x-y) - K(x)|dx \le C, \quad \forall y \in \mathbb{R}^n. \tag{2.2.3}$$

(b) If K satisfies Hörmander condition (2.2.3), then $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1})$ and Ω satisfies L^1-Dini condition.

The proof of Theorem 2.2.1 can be found in Calderón- Weiss- Zygmund [CaWZ] and Calderón- Zygmund [CaZ4].

The following consequence improves Theorem 2.1.2.

Theorem 2.2.2 *Suppose that Ω satisfies (2.2.1), (2.2.2) and L^1-Dini condition. Let*

$$T_\Omega f(x) = \text{p.v.} \int \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$$

for $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$. Then

(i) T_Ω is of type (p,p) $(1 < p < \infty)$;

(ii) T_Ω is of weak type (1,1).

First we formulate a lemma which will be used in the proof of Theorem 2.2.2.

Lemma 2.2.1 *Suppose that $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then, for any ε, $0 < \varepsilon < 1/2$, there exists a constant $C > 0$ such that*

$$\int_{2^j}^{2^{j+1}} \left| \int_{\mathbb{S}^{n-1}} \Omega(y') e^{-2\pi i r x' \cdot y'} d\sigma(y') \right| \frac{dr}{r} \le C \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} 2^{-j\varepsilon} \tag{2.2.4}$$

holds for every $j \in \mathbb{N}$ and $x' \in \mathbb{S}^{n-1}$.

Proof. By Hölder's inequality, the left side of (2.2.4) is dominated by

$$(\log 2)^{\frac{1}{2}}\left(\int_{2^j}^{2^{j+1}}\left|\int_{\mathbb{S}^{n-1}}\Omega(y')e^{-2\pi i r x'\cdot y'}d\sigma(y')\right|^2\frac{dr}{r}\right)^{\frac{1}{2}}$$

$$=(\log 2)^{\frac{1}{2}}\left(\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\Omega(y')\overline{\Omega(u')}\int_{2^j}^{2^{j+1}}e^{-2\pi i r x'\cdot(y'-u')}\frac{dr}{r}d\sigma(y')d\sigma(u')\right)^{\frac{1}{2}}.$$

Set

$$I=\int_{2^j}^{2^{j+1}}e^{-2\pi i r x'\cdot(y'-u')}\frac{dr}{r}.$$

Clearly, we have $|I|\le\log 2$. On the other hand, we obtain

$$|I|\le C\left|2^j x'\cdot(y'-u')\right|^{-1},$$

where C is independent of j, x', y' and u'. Hence, for any α, $0<\alpha<1$, it follows that

$$|I|\le C\min\left\{1,\left|2^j x'\cdot(y'-u')\right|^{-1}\right\}\le C\left|2^j x'\cdot(y'-u')\right|^{-\alpha}. \qquad (2.2.5)$$

Since $0<\alpha<1$, there exists a constant C_1 independent of x', such that

$$\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\frac{1}{|x'\cdot(y'-u')|^{\alpha}}d\sigma(y')d\sigma(u')\le C_1.$$

Now let $\varepsilon=\frac{\alpha}{2}$, then the above inequality together with (2.2.5) implies (2.2.4). ∎

Next we turn to the proof of Theorem 2.2.2. First consider the case $p=2$. Set $K(x)=\Omega(x)|x|^{-n}$. Then $T_\Omega f=\text{p.v.}\,K*f$. We now give the estimate of $\hat{K}(\xi)$. Let $\varphi\in\mathscr{S}(\mathbb{R}^n)$. Then it follows that

$$\begin{aligned}\langle\widehat{K},\varphi\rangle&=\langle K,\hat{\varphi}\rangle\\&=\lim_{\varepsilon\to 0,N\to\infty}\int_{\varepsilon\le|x|\le N}K(x)\int_{\mathbb{R}^n}e^{-2\pi i x\cdot y}\varphi(y)dydx\\&=\lim_{\varepsilon\to 0,N\to\infty}\int_{\mathbb{R}^n}\varphi(y)\int_{\varepsilon\le|x|\le N}K(x)e^{-2\pi i x\cdot y}dxdy\\&=\lim_{\varepsilon\to 0,N\to\infty}\int_{\mathbb{R}^n}\varphi(y)\int_{\varepsilon/|y|}^{N/|y|}\int_{\mathbb{S}^{n-1}}\Omega(x')e^{-2\pi i r x'\cdot y'}d\sigma(x')\frac{dr}{r}dy.\end{aligned}$$

Clearly, if we can show that there exists a constant C independent of y', such that

$$\int_0^\infty \left| \int_{\mathbb{S}^{n-1}} \Omega(x') e^{-2\pi i r x'\cdot y'} d\sigma(x') \right| \frac{dr}{r} \leq C, \tag{2.2.6}$$

then the dominated convergence theorem implies $|\widehat{K}(y)| \leq C$ for every $y \in \mathbb{R}^n$). Consequently T_Ω is an operator of type $(2,2)$. Now set

$$\begin{aligned} E_0 &= \{x' \in \mathbb{S}^{n-1} : |\Omega(x')| < 2\} \\ E_k &= \{x' \in \mathbb{S}^{n-1} : 2^k \leq |\Omega(x')| < 2^{k+1}\}, \qquad k \in \mathbb{N} \end{aligned}$$

and

$$\Omega_k(x') = \Omega(x')\chi_{E_k}(x') \qquad (k \geq 0).$$

Since Ω satisfies the vanishing condition, it implies that

$$\int_0^2 \left| \int_{\mathbb{S}^{n-1}} \Omega(x') \left(e^{-2\pi i r x'\cdot y'} - 1 \right) d\sigma(x') \right| \frac{dr}{r} \leq C \|\Omega\|_{L^1(\mathbb{S}^{n-1})}.$$

While

$$\begin{aligned} &\int_2^\infty \left| \int_{\mathbb{S}^{n-1}} \Omega(x') e^{-2\pi i r x'\cdot y'} d\sigma(x') \right| \frac{dr}{r} \\ &\leq \sum_{j=1}^\infty \int_{2^j}^{2^{j+1}} \sum_{k=0}^\infty \left| \int_{\mathbb{S}^{n-1}} \Omega_k(x') e^{-2\pi i r x'\cdot y'} d\sigma(x') \right| \frac{dr}{r} \\ &= \sum_{j=1}^\infty \int_{2^j}^{2^{j+1}} \left| \int_{\mathbb{S}^{n-1}} \Omega_0(x') e^{-2\pi i r x'\cdot y'} d\sigma(x') \right| \frac{dr}{r} \\ &\quad + \sum_{j=1}^\infty \sum_{k=1}^\infty \int_{2^j}^{2^{j+1}} \left| \int_{\mathbb{S}^{n-1}} \Omega_k(x') e^{-2\pi i r x'\cdot y'} d\sigma(x') \right| \frac{dr}{r} \\ &:= I_1 + I_2. \end{aligned}$$

Thus applying (2.2.4) leads to

$$I_1 \leq \sum_{j=1}^\infty C \cdot 2^{-j\varepsilon} \leq C.$$

Now take $l \in \mathbb{N}$ such that $l\varepsilon > 1$. We have that

$$\begin{aligned}I_2 =&\sum_{k=1}^{\infty}\sum_{1\le j\le lk}\int_{2^j}^{2^{j+1}}\left|\int_{\mathbb{S}^{n-1}}\Omega_k(x')e^{-2\pi i r x'\cdot y'}d\sigma(x')\right|\frac{dr}{r}\\&+\sum_{k=1}^{\infty}\sum_{j>lk}\int_{2^j}^{2^{j+1}}\left|\int_{\mathbb{S}^{n-1}}\Omega_k(x')e^{-2\pi i r x'\cdot y'}d\sigma(x')\right|\frac{dr}{r}\\:=&I_{21}+I_{22}.\end{aligned}$$

Applying (2.2.4), we obtain

$$I_{22} \le C\sum_{k=1}^{\infty}\sum_{j>lk}\|\Omega_k\|_{L^\infty(\mathbb{S}^{n-1})}2^{-j\varepsilon} < \infty,$$

while

$$I_{21} \le \sum_{k=1}^{\infty}\sum_{1\le j\le lk}2^{k+1}\cdot|E_k|\cdot\log 2 \le C\|\Omega\|_{L\log^+L(\mathbb{S}^{n-1})} < \infty.$$

Thus from the estimates of I_1, I_{21} and I_{22}, it follows (2.2.6). Therefore T_Ω is an operator of type $(2, 2)$.

Now we illustrate that T_Ω is of weak type (1, 1). In fact, by the condition (a) in Theorem 2.2.1, we know that $K(x) = \Omega(x)|x|^{-n}$ satisfies Hörmander condition (2.2.3). This, together with that T_Ω is of type $(2, 2)$, concludes that T_Ω is of weak type $(1, 1)$ (See the proof of Theorem 2.1.1). Finally, applying the Marcinkiewicz interpolation theorem and dual method yields that T_Ω is of type (p, p) $(1 < p < \infty)$.

Remark 2.2.2 *As stated above, when $\Omega \in L\log^+L(\mathbb{S}^{n-1})$ and satisfies the vanishing condition, we can get L^2-boundedness of T_Ω without requiring that Ω satisfies L^1-Dini condition. In fact, later we will obtain that T_Ω is of type $(p,p)(1 < p < \infty)$ and weak type(1, 1), provided that $\Omega \in L\log^+L(\mathbb{S}^{n-1})$ and satisfies the vanishing condition* (2.2.2). *(See Theorem 2.3.4 and Remark 2.3.1).*

Now we start to discuss the weighted L^p-boundedness of T_Ω. By Theorem 2.1.6, if Ω satisfies the homogeneity of degree 0 as well as the vanishing condition, then L^∞-Dini condition (2.1.9) can ensure T_Ω is bounded from $L^p(\omega)$ to $L^p(\omega)$ for $1 < p < \infty$ and $\omega \in A_p$. As to the conclusion of Theorem 2.2.2, a natural question is: if we weaken the L^∞-Dini condition to L^1-Dini condition with $1 < p < \infty$ and $\omega \in A_p$, can we still ensure T_Ω is bounded from $L^p(\omega)$ to $L^p(\omega)$? However, there are counterexamples to illustrate that conclusion would not hold even $\omega \in A_1$. So in the following we consider the L^q-Dini condition $(1 < q < \infty)$ instead of the smooth condition of Ω. We will also discuss the weighted L^p-boundedness of the corresponding singular integral operator T_Ω. Obviously we may predict that, in this situation, the weight function ω will depend not only on p, but also on q.

We have the following statement.

Theorem 2.2.3 *Suppose that Ω satisfies* (2.2.1), (2.2.2) *and L^q-Dini condition* $(1 < q < \infty)$. *When either of the following two conditions (i) or (ii) holds, T_Ω is bounded from $L^p(\omega)$ to $L^p(\omega)$.*

(i) $q' \le p < \infty$ *and* $\omega \in A_{p/q'}$;

(ii) $1 < p \le q$ *and* $\omega^{-1/(p-1)} \in A_{p'/q'}$.

In addition, if $\omega^{q'} \in A_1$, then T_Ω is bounded from $L^1(\omega)$ to $L^{1,\infty}(\omega)$, i.e., there exits a constant $C > 0$ such that

$$\omega\left(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}\right) \le \frac{C}{\lambda}\|f\|_{L^1(\omega)}$$

for every $\lambda > 0$ and $f \in L^1(\omega)$.

The proof of Theorem 2.2.3 will be obtained by applying the idea in the proof of Theorem 2.1.6. We first give some lemmas.

Lemma 2.2.2 *Suppose that Ω satisfies (2.2.1) and L^q-Dini condition* $(1 \le q < \infty)$. *Then for any $R > 0$ and $x \in \mathbb{R}^n$, when $|x| < R/2$, there exists a constant $C > 0$ such that*

$$\left(\int_{R<|y|\le 2R}\left|\frac{\Omega(y-x)}{|x-y|^n} - \frac{\Omega(y)}{|y|^n}\right|^q dy\right)^{\frac{1}{q}} \le CR^{\frac{-n}{q'}}\left\{\frac{|x|}{R} + \int_{\frac{|x|}{R}}^{\frac{2|x|}{R}} \frac{\omega_q(\delta)}{\delta}d\delta\right\}. \tag{2.2.7}$$

Proof. Since $|x-y| \sim |y|$, we have

$$\left|\frac{\Omega(y-x)}{|x-y|^n} - \frac{\Omega(y)}{|y|^n}\right| \le C\left\{|\Omega(y)|\frac{|x|}{|y|^{n+1}} + \frac{|\Omega(y-x) - \Omega(y)|}{|y|^n}\right\}.$$

Hence

$$
\begin{aligned}
&\left(\int_{R<|y|\le 2R}\left|\frac{\Omega(y-x)}{|x-y|^n}-\frac{\Omega(y)}{|y|^n}\right|^q dy\right)^{1/q}\\
&\le C\left(\int_{R<|y|\le 2R}|\Omega(y)|^q\frac{|x|^q}{|y|^{(n+1)q}}dy\right)^{1/q}\\
&\quad + C\left(\int_{R<|y|\le 2R}\frac{|\Omega(y-x)-\Omega(y)|^q}{|y|^{nq}}dy\right)^{1/q}\\
&:= I_1+I_2.
\end{aligned}
$$

Since $\Omega\in L^q(\mathbb{S}^{n-1})$, it follows that

$$
I_1\le C\|\Omega\|_{L^q(\mathbb{S}^{n-1})}|x|\cdot R^{-(n+1)}\cdot R^{n/q}=CR^{-n/q'}\cdot\frac{|x|}{R}\ .
$$

On the other hand, we have that

$$
\begin{aligned}
I_2&\le C\left(\int_R^{2R}t^{-nq+n-1}\int_{\mathbb{S}^{n-1}}|\Omega(ty'-x)-\Omega(y')|^q d\sigma(y')dt\right)^{1/q}\\
&\le CR^{-n/q'}\left(\int_R^{2R}\int_{\mathbb{S}^{n-1}}\left|\Omega\left(\frac{y'-\alpha}{|y'-\alpha|}\right)-\Omega(y')\right|^q d\sigma(y')\frac{dt}{t}\right)^{1/q},
\end{aligned}
$$

where $\alpha=\frac{x}{t}$. Applying a result of Calderòn- Weiss-Zygmund (see [CaWZ]) to $|\alpha|=\frac{|x|}{t}<\frac{1}{2}$ and $1\le q<\infty$, we conclude that

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}}\left|\Omega\left(\frac{y'-\alpha}{|y'-\alpha|}\right)-\Omega(y')\right|^q d\sigma(y')&\le C\sup_{\|\rho\|<\alpha}\int_{\mathbb{S}^{n-1}}|\Omega(\rho x')-\Omega(x')|^q d\sigma(x')\\
&\le C\omega_q^q\left(\frac{|x|}{t}\right)\\
&\le C\omega_q^q\left(\frac{|x|}{R}\right).
\end{aligned}
$$

Here we use the property that $\omega(\delta)$ is nondecreasing in δ. Thus,

$$
\begin{aligned}
I_2&\le CR^{-n/q'}\cdot\omega_q\left(\frac{|x|}{R}\right)\cdot(\log 2)^{1/q}\\
&\le C'R^{-n/q'}\cdot\omega_q\left(\frac{|x|}{R}\right)\cdot\int_{|x|/R}^{2|x|/R}\frac{d\delta}{\delta}\\
&\le C'R^{-n/q'}\int_{|x|/R}^{2|x|/R}\frac{\omega_q(\delta)}{\delta}d\delta.
\end{aligned}
$$

So (2.2.7) follows. ∎

Lemma 2.2.3 *Suppose that* Ω *satisfies* (2.2.1) *and* L^q*-Dini condition* $(1 < q < \infty)$. *Then there exists a constant* $C > 0$ *such that*

$$\left| \int_{|y-x_0|>4t} \left(\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x_0-y)}{|x_0-y|^n} \right) f(y)dy \right| \le C \left[M\left(|f|^{q'} \right)(x_0) \right]^{1/q'} \tag{2.2.8}$$

for any $x_0 \in \mathbb{R}^n, t > 0$ *and every* x *with* $|x - x_0| < t$, *where* M *is the Hardy-Littlewood maximal function.*

Proof. From Hölder's inequality and (2.2.7) it follows that

$$\begin{aligned}
&\left| \int_{|y-x_0|>4t} \left(\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x_0-y)}{|x_0-y|^n} \right) f(y)dy \right| \\
\le& \sum_{j=2}^{\infty} \left(\int_{2^j t<|y-x_0|\le 2^{j+1}t} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x_0-y)}{|x_0-y|^n} \right|^q dy \right)^{1/q} \\
&\quad \times \left(\int_{|y-x_0|\le 2^{j+1}t} |f(y)|^{q'} dy \right)^{1/q'} \\
\le& C \sum_{j=2}^{\infty} \left(\frac{1}{2^j} + \int_{|x_0-x|/2^j t}^{|x_0-x|/2^{j-1}t} \frac{\omega_q(\delta)}{\delta} d\delta \right) \left(\frac{1}{(2^{j+1}t)^n} \int_{|y-x_0|\le 2^{j+1}t} |f(y)|^{q'} dy \right)^{1/q'} \\
\le& C \left(M\left(|f|^{q'} \right)(x_0) \right)^{1/q'} \left(1 + \int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta \right) \\
\le& C \left(M\left(|f|^{q'} \right)(x_0) \right)^{1/q'}.
\end{aligned}$$

∎

Lemma 2.2.4 *Suppose that* Ω *satisfies* (2.2.1) *and* L^q*-Dini condition* $(1 < q < \infty)$. *Then there exists a constant* $C = C(n, q)$ *such that*

$$M^\sharp (T_\Omega f)(x) \le C(n,q) \left(M\left(|f|^{q'} \right)(x) \right)^{1/q'}, \quad x \in \mathbb{R}^n, \tag{2.2.9}$$

where $M^\sharp$ *is the sharp maximal function defined in the section 2.1.*

Proof. The idea of proof is similar to that of Lemma 2.1.2. Choose $x \in \mathbb{R}^n$ and a cube Q containing x. Let B be a ball with the same center and diameter as that of Q. Set $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{16B}$. Theorem 2.2.2 yields that

$$\frac{1}{|Q|}\int_Q |T_\Omega f_1(y)|dy \le C\left(\frac{1}{|Q|}\int_{16B}|f(y)|^{q'}dy\right)^{1/q'} \le C\left(M\left(|f|^{q'}\right)(x)\right)^{1/q'}.$$

On the other hand, denote the diameter of B by d. Then applying (2.2.8) implies that

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |T_\Omega f_2(y) - T_\Omega f_2(x)|dy\\ =&\frac{1}{|Q|}\int_Q \left|\int_{\mathbb{R}^n\setminus 16B}\left[\frac{\Omega(y-z)}{|y-z|^n} - \frac{\Omega(x-z)}{|x-z|^n}\right] f(z)dz\right| dy\\ \le& C\left[M\left(|f|^{q'}\right)(x)\right]^{1/q'}.\end{aligned}$$

From (2.1.19), Lemma 2.2.4 follows. ■

To prove Theorem 2.2.3, we still need a celebrated interpolation theorem.

Lemma 2.2.5 (Stein-Weiss interpolation theorem with change of measure) *Suppose that u_0, v_0, u_1, v_1 are positive weight functions and $1 < p_0,\ p_1 < \infty$. Assume sublinear operator S satisfies :*

$$\|Sf\|_{L^{p_0}(u_0)} \le C_0\|f\|_{L^{p_0}(v_0)}$$

and

$$\|Sf\|_{L^{p_1}(u_1)} \le C_1\|f\|_{L^{p_1}(v_1)}.$$

Then

$$\|Sf\|_{L^p(u)} \le C\|f\|_{L^p(v)}$$

holds for any $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1},$$

where

$$u = u_0^{p\theta/p_0}u_1^{p(1-\theta)/p_1}, \quad v = v_0^{p\theta/p_0}v_1^{p(1-\theta)/p_1}$$

and $C \le C_0^\theta C_1^{1-\theta}$.

For the proof of Lemma 2.2.5, see Stein and Weiss [StW].

Proof of Theorem 2.2.3. First consider the case which the condition (i) holds. Analogous to the proof of Theorem 2.1.6, we merely need to discuss the bounded function with compact support. Assume f will be the case. We need to show $M(T_\Omega f)(x) \in L^p(\omega)$, provided that $q' < p < \infty$ and $\omega \in A_{p/q'}$. By the L^p-boundedness of M, it suffices to verify $T_\Omega f \in L^p(\omega)$. We may suppose $\mathrm{supp}(f) \subset \{x \in \mathbb{R}^n : |x| < R\}$ as well. Set

$$\begin{aligned}\int_{\mathbb{R}^n} |T_\Omega f(x)|^p \omega(x)dx &= \int_{|x|\le 2R} |T_\Omega f(x)|^p \omega(x)dx + \int_{|x|>2R} |T_\Omega f(x)|^p \omega(x)dx \\ &:= I_1 + I_2.\end{aligned}$$

By the $L^r(1 < r < \infty)$-boundedness of T_Ω (Theorem 2.2.2) and the Reverse Hölder inequality, we see that I_1 is finite. Thus we mainly consider I_2. Note that $|y| < R$ and $|x| > 2R$. it implies that $|x - y| \sim |x|$. Consequently, it follows that

$$\begin{aligned}I_2 &\le C\|f\|^p_{L^\infty} \int_{|x|>2R} \left(\int_{|y|<R} |\Omega(x-y)|dy\right)^p \frac{\omega(x)}{|x|^{np}}dx \\ &\le C' \int_{|x|>2R} \left(\int_{|x-y|<\frac{3}{2}|x|} |\Omega(x-y)|^q dy\right)^{p/q} \frac{\omega(x)}{|x|^{np}}dx \\ &\le C'' \int_{|x|>2R} \frac{\omega(x)}{|x|^{np-np/q}}dx \\ &= C'' \sum_{j=1}^{\infty} \int_{2^j R<|x|\le 2^{j+1}R} \frac{\omega(x)}{|x|^{np/q'}}dx \\ &\le C'' \sum_{j=1}^{\infty} (2^j R)^{-np/q'} \omega(B(0, 2^{j+1}R)).\end{aligned}$$

Since $\omega \in A_{p/q'}$, there exists $1 < s < p/q'$ such that $\omega \in A_s$. Hence applying (2.1.33) yields that

$$\omega(B(0, 2^{j+1}R)) \le C \cdot (2^{j+1}R)^{ns} \omega(B(0,1)),$$

where C is independent of j. So we obtain that

$$I_2 \le C'' \sum_{j=1}^{\infty} (2^j R)^{-n(p/q'-s)} \omega(B(0,1)) < \infty,$$

which implies that $M(T_\Omega f) \in L^p(\omega)$. Applying Lemma 2.1.3, Lemma 2.2.4 and the weighted boundedness of M (Theorem 1.4.3), we conclude that, for a bounded function f with compact support,

$$\begin{aligned}\int_{\mathbb{R}^n} |T_\Omega f(x)|^p \omega(x)dx &\le \int_{\mathbb{R}^n} [M(T_\Omega f)(x)]^p\, \omega(x)dx \\ &\le C\int_{\mathbb{R}^n} [M^\sharp(T_\Omega f)(x)]^p \omega(x)dx \\ &\le C\int_{\mathbb{R}^n} \left[M(|f|^{q'})(x)\right]^{p/q'} \omega(x)dx \\ &\le C'\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx.\end{aligned}$$

When $q' < p < \infty$ and $\omega \in A_{p/q'}$, T_Ω is bounded from $L^p(\omega)$ to $L^p(\omega)$. Consider the case $p = q'$, clearly $\omega \in A_1$. Thus there exists $p_1 > p = q'$ such that $\omega^{p_1/p} \in A_1 \subset A_{p_1/q'}$. Thus there exists $\varepsilon > 0$ such that

$$\omega^{p_1(1+\varepsilon)/p} \in A_{p_1/q'}.$$

Now set $\theta = \frac{\varepsilon}{1+\varepsilon}$ and choose p_0 such that

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Clearly $1 < p_0 < p$. First we have

$$\|T_\Omega f\|_{L^{p_0}} \le C_0 \|f\|_{L^{p_0}}.$$

On the other hand, the above conclusion implies that

$$\|T_\Omega f\|_{L^{p_1}(u_1)} \le C_1 \|f\|_{L^{p_1}(u_1)},$$

where

$$u_1 = \omega^{p_1(1+\varepsilon)/p} \in A_{p_1/q'}.$$

It follows that T_Ω is bounded from $L^p(\omega)$ to $L^p(\omega)$ by applying Lemma 2.2.5 to $u_0 = v_0 = 1$ and $u_1 = v_1 = \omega^{p_1(1+\varepsilon)/p}$. Up to now, we have shown that T_Ω is bounded from $L^p(\omega)$ to $L^p(\omega)$ under the condition (i).

Now we discuss the case which the condition (ii) holds. Let $\widetilde{T_\Omega}$ be the conjugate operator of T_Ω. Then we have

$$\widetilde{T_\Omega} f(x) = T_{\widetilde{\Omega}} f(x),$$

with $\widetilde{\Omega}(x) = \overline{\Omega(-x)}$. Clearly $\widetilde{\Omega}$ satisfies all the conditions on Ω. Consequently, for any $f \in L^p(\omega)(1 < p \le q)$, we have

$$\|T_\Omega f\|_{L^p(\omega)} = \sup \left| \int_{\mathbb{R}^n} T_\Omega f(x) \cdot g(x) dx \right|,$$

where the supremum is taken over all functions $g \in L^{p'}(\omega^{-1/(p-1)})$ with $\|g\|_{L^{p'}(\omega^{-1/(p-1)})} \le 1$. Clearly $q' \le p' < \infty$ follows from $1 < p \le q$. By $\omega^{-1/(p-1)} \in A_{p'/q'}$ together with the above conclusion we know that $\widetilde{T}_\Omega$ is bounded from $L^{p'}(\omega^{-1/(p-1)})$ to $L^{p'}(\omega^{-1/(p-1)})$. Thus

$$\begin{aligned} \|T_\Omega f\|_{L^p(\omega)} &= \sup \left| \int_{\mathbb{R}^n} f(x) \widetilde{T}_\Omega g(x) dx \right| \\ &\le \|f\|_{L^p(\omega)} \sup \|\widetilde{T}_\Omega g\|_{L^{p'}(\omega^{-1/(p-1)})} \\ &\le C \|f\|_{L^p(\omega)}. \end{aligned}$$

Therefore this completes the proof of Theorem 2.2.3 under the condition (ii).

Finally we wish to show that T_Ω is bounded from $L^1(\omega)$ to $L^{1,\infty}(\omega)$ as $\omega^{q'} \in A_1$. Since the idea is similar to that of Theorem 2.1.7, we only give the differentia. For any $\lambda > 0$, we make the Calderón-Zygmund decomposition: $f = g + b$, where g is the "good" function and b is the "bad" one. If $\omega^{q'} \in A_1$, then $\omega \in A_1$. The estimate of I_1 follows from applying the boundedness of T_Ω on $L^{q'}(\omega)$. The estimate of I_2 can be obtained by applying Lemma 2.2.3 and $\omega^{q'} \in A_1$. Here we omit the details. We have finished the proof of Theorem 2.2.3.

In the proof of Theorem 2.2.3 we see that $\omega^{q'} \in A_1$ implies that T_Ω is of weighted weak type (1,1). The following conclusion shows that $\omega^{q'} \in A_p (1 < p < \infty)$ is also a sufficient condition for T_Ω to be of weighted type (p, p).

Theorem 2.2.4 *Suppose that Ω satisfies* (2.2.1), (2.2.2) *and L^q-Dini condition* $(1 < q < \infty)$. *If $\omega^{q'} \in A_p$, then T_Ω is bounded from $L^p(\omega)$ to $L^p(\omega)$ for $1 < p < \infty$.*

Proof. We will use Theorem 2.2.3 and Lemma 2.2.5 to prove Theorem 2.2.4. For the purpose, we have to verify that there exist θ $(0 < \theta < 1)$, p_0, p_1 and weight function ω_0, ω_1 satisfying the following conditions:

(i) $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$ and $1 < p_0 < \min\{q, p\}$, $\max\{q', p\} < p_1 < \infty$;

(ii) $\omega = \omega_0^{p\theta/p_0} \omega_1^{p(1-\theta)/p_1}$ and $\omega_0^{1-p_0'} \in A_{p_0'/q'}$, $\omega_1 \in A_{p_1/q'}$.

Since $\omega^{q'} \in A_p$ $(1 < p < \infty)$, applying the Jones decomposition theorem of A_p weight (Theorem 1.4.5), there exist $u, v \in A_1$ such that

$$\omega^{q'} = u \cdot v^{1-p}, \quad \text{i.e.,} \quad \omega = u^{1/q'} v^{(1-p)/q'}.$$

Set

$$\omega = (u^\alpha v^\beta)^s (u^\gamma v^\delta)^{1-s},$$

then we have

$$\alpha s + \gamma(1-s) = \frac{1}{q'} \tag{2.2.10}$$

and

$$\beta s + \delta(1-s) = \frac{1-\rho}{q'}. \tag{2.2.11}$$

Put $\omega_0 = u^\alpha v^\beta$ and $\omega_1 = u^\gamma v^\delta$. First we illustrate that

$$\omega_0^{1-p_0'} \in A_{p_0'/q'}$$

if $1 < p_0 < \min\{q, p\}$ and $\alpha = [(p_0'/q') - 1]/(p_0' - 1)$, $\beta = 1/(1 - p_0')$. In fact, since $u, v \in A_1$, we have that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \omega_0(x)^{1-p_0'} dx\right)\left(\frac{1}{|Q|}\int_Q \omega_0(x)^{-(1-p_0')/[(p_0'/q')-1]} dx\right)^{(p_0'/q')-1} \\
&= \left(\frac{1}{|Q|}\int_Q u(x)^{\alpha(1-p_0')} v(x)^{\beta(1-p_0')} dx\right) \\
&\quad \times \left(\frac{1}{|Q|}\int_Q u(x)^{-\alpha(1-p_0')/[(p_0'/q')-1]} v(x)^{-\beta(1-p_0')/[(p_0'/q')-1]} dx\right)^{(p_0'/q')-1} \\
&\le C\left(\frac{1}{|Q|}\int_Q u(x) dx\right)^{\alpha(1-p_0')}\left(\frac{1}{|Q|}\int_Q v(x) dx\right)^{\beta(1-p_0')} \\
&\quad \times \left(\frac{1}{|Q|}\int_Q v(x) dx\right)^{-\beta(1-p_0')}\left(\frac{1}{|Q|}\int_Q u(x)^{-\alpha(1-p_0')/[(p_0'/q')-1]} dx\right)^{(p_0'/q')-1} \\
&\le C.
\end{aligned}$$

Here C is independent of Q. In the same way we can prove

$$\omega_1 \in A_{p_1/q'}$$

if $\max\{q', p\} < p_1 < \infty$ and $\gamma = 1$, $\delta = 1 - p_1/q'$. Thus the value of α and γ, together with the equality (2.2.10), imply $s = 1/p_0$. Similarly, the value of β and δ, together with $s = \frac{1}{p_0}$, lead to $p_1 = p_0'(p-1)$ by invoking the equality (2.2.11). Set $\theta = p_0(p_1 - p)/p(p_1 - p_0)$. Then we have $0 < \theta < 1$ and $s = p\theta/p_0, 1 - s = p(1-\theta)/p_1$. Thus by our choice of $p_0,\ p_1,\ \theta,\ \omega_0,\ \omega_1$ we know both (i) and (ii) hold. Finally, we obtain Theorem 2.2.4. ■

Remark 2.2.3

1. When $q = \infty$, Theorem 2.2.3 and Theorem 2.2.4 are in accordance with Theorem 2.1.6 and Theorem 2.1.7 respectively.

2. In Section 2.1 we call K is the Calderón-Zygmund kernel (see 2.1.1), if K satisfies Hörmander condition (2.1.3). Actually, we can also define L^q-Hörmander condition H_q $(1 \le q \le \infty)$. If $K \in L^1_{loc}(\mathbb{R}\backslash\{0\})$ and there exist $C, C_r > 0$, such that for any $y \in \mathbb{R}^n$ and $R > C|y|$, the following condition

$$(H_q) \quad \sum_{k=1}^{\infty} \left(2^k R\right)^{n/q'} \left(\int_{2^k R<|x|\le 2^{k+1}R} |K(x-y) - K(x)|^q dx\right)^{1/q} \le C_r$$

and

$$(H_\infty) \quad \sum_{k=1}^{\infty} \left(2^k R\right)^{n} \sup_{2^k R<|x|\le 2^{k+1}R} |K(x-y) - K(x)| \le C_\infty$$

hold. Sometimes (H_q) and (H_∞) are called L^q-Hörmander conditions $(1 \le q \le \infty)$. Clearly, (2.1.3) is just condition (H_1).

By the proof of Lemma 2.2.2 and (2.1.4) we know that if $K(x) = \frac{\Omega(x)}{|x|^n}$ and Ω satisfies (2.2.1) as well as L^q-Dini condition $(1 \le q \le \infty)$, then K must satisfy the condition (H_q) $(1 \le q \le \infty)$. In this sense, the condition (H_q) $(1 \le q \le \infty)$ is weaker than L^q-Dini condition.

2.3 Singular integral operators with rough kernels

In the last section we have seen, for the singular integral operator T_Ω with homogeneous kernel, that Ω satisfies L^1-Dini condition can ensure (p,p) boundedness and weak (1,1) boundedness of T_Ω, whenever Ω has the homogeneity of degree 0 and satisfies the vanishing condition. The L^1-Dini condition is much weaker than L^∞-Dini condition. However, the L^1-Dini condition still involves certain smoothness of Ω on the unit sphere. On the other hand, from the proof of Theorem 2.2.2 we note that L^1-Dini condition was not used in the proof of the $(2,2)$ boundedness of T_Ω (see Remark 2.2.2). This fact suggests us whether or not we can still keep the (p,p) boundedness and weak (1,1) boundedness of T_Ω without any smoothness of Ω on the unit sphere. It is just the question that we will discuss in this section.

We begin with discussing the rotation method and singular integral operators with rough odd kernel. In 1956, Calderon and Zygmund [CaZ3] first considered the above problem, and introduced "rotation method" to deal with the L^p-boundedness ($1 < p < \infty$) of a class of singular integral operators with rough odd kernels. In short, the rotation method, by sphere coordinate transform, is to reduce the study of convolution operator on $\mathbb{R}^n$ to the study of an operator in one dimension in some direction.

For any $y \in \mathbb{R}^n, y \neq 0$, we write $y = ry'$, where $0 < r < \infty$ and $y' \in \mathbb{S}^{n-1}$. Set $Tf(x) = K * f(x)$, we have

$$Tf(x) = \int_{\mathbb{R}^n} K(y)f(x-y)dy = \int_{\mathbb{S}^{n-1}} \int_0^\infty K(ry')f(x-ry')r^{n-1}drd\sigma(y').$$

Set $K(ry') = \omega(y')h(r)$. We have

$$\begin{aligned} Tf(x) &= \int_{\mathbb{S}^{n-1}} \omega(y') \int_0^\infty h(r)f(x-ry')r^{n-1}drd\sigma(y') \\ &:= \int_{\mathbb{S}^{n-1}} \omega(y')T_{y'}f(x)d\sigma(y'). \end{aligned}$$

Now for fixed $y' \in \mathbb{S}^{n-1}$, let Y be the hyperplane through the origin orthogonal to y'. Then for all $x \in \mathbb{R}^n$, there exist $s \in \mathbb{R}$ and $z \in Y$, such that $x = z + sy'$. Hence

$$T_{y'}f(x) = \int_0^\infty h(r)f[z+(s-r)y']r^{n-1}dr := L(f_{z,y'})(s).$$

If L is of type (p,p) ($1 < p < \infty$) on $\mathbb{R}^1$, then it follows that

$$\begin{aligned} \int_{\mathbb{R}^n} \left|T_{y'}f(x)\right|^p dx &= \int_Y \int_{-\infty}^\infty \left|L(f_{z,y'})(s)\right|^p dsdz \\ &\le C\int_Y \int_{-\infty}^\infty \left|f(z+sy')\right|^p dsdz \\ &= C\int_{\mathbb{R}^n} |f(x)|^p dx, \end{aligned}$$

where C is independent of z, y' and f. Thus Minkowski's inequality yields

that

$$\begin{aligned}\|Tf\|_p &= \left\| \int_{\mathbb{S}^{n-1}} \omega(y')T_{y'}f(\cdot)d\sigma(y') \right\|_p \\ &\le \int_{\mathbb{S}^{n-1}} |\omega(y')| \cdot \|T_{y'}f\|_p d\sigma(y') \\ &\le C\|f\|_p \cdot \int_{\mathbb{S}^{n-1}} |\omega(y')|d\sigma(y').\end{aligned}$$

Therefore when

$$\int_{\mathbb{S}^{n-1}} |\omega(y')|d\sigma(y') < \infty,$$

L^p-boundedness of T follows immediately.

Now using the above idea, we will obtain the boundedness of a class of singular integral operators with homogeneous kernels.

Theorem 2.3.1 *Suppose that* Ω *satisfies* (2.2.1) *and* $\Omega(-x') = -\Omega(x')$, *for every* $x' \in \mathbb{S}^{n-1}$. *If* $\Omega \in L^1(\mathbb{S}^{n-1})$, *then* T_Ω *is of type* (p,p) *for* $1 < p < \infty$.

Proof. Using the rotation method, we have that

$$\begin{aligned}T_\Omega f(x) &= \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)dy \\ &= \frac{\pi}{2} \int_{\mathbb{S}^{n-1}} \Omega(y') \left(\frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(x-ry')}{r} \right) d\sigma(y') \\ &:= \frac{\pi}{2} \int_{\mathbb{S}^{n-1}} \Omega(y') H_{y'}f(x)d\sigma(y'),\end{aligned}$$

where $H_{y'}$ is the Hilbert transform along y' direction. With the above idea and (p,p) boundedness of the Hilbert transform (Theorem 2.1.5), and also noticing $\Omega \in L^1(\mathbb{S}^{n-1})$, we immediately deduce that T_Ω is of type (p,p). ■

Similarly, we can define the maximal Hilbert transform $H^*_{y'}$ and the Hardy-Littlewood maximal operator $M_{y'}$ along y' direction:

$$\begin{aligned}H^*_{y'}f(x) &= \frac{1}{\pi} \sup_{\varepsilon>0} \left| \int_{|t|>\varepsilon} \frac{f(x-ty')}{t} dt \right|, \\ M_{y'}f(x) &= \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x-ty')|dt.\end{aligned}$$

Using the rotation method we get the following result.

Theorem 2.3.2 *Suppose that* Ω *satisfies* (2.2.1) *and* $\Omega(-x') = -\Omega(x')$, *for every* $x' \in \mathbb{S}^{n-1}$. *If* $\Omega \in L^1(\mathbb{S}^{n-1})$, *then* T_Ω^* *is of type* (p,p) *for* $1 < p < \infty$.

It follows that

$$\begin{aligned} T_\Omega^* f(x) &= \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy \right| \\ &\le \frac{\pi}{2} \int_{\mathbb{S}^{n-1}} |\Omega(y')| H_{y'}^* f(x) d\sigma(y'). \end{aligned}$$

By the rotation method and Remark 2.1.5, Theorem 2.3.2 is easily proved.

Now define the maximal operator M_Ω with rough kernel by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |\Omega(y)||f(x-y)|dy. \tag{2.3.1}$$

It is a very important tool in the study of singular integral operators. Since

$$M_\Omega f(x) \le \int_{\mathbb{S}^{n-1}} |\Omega(y')| M_{y'} f(x) d\sigma(y'),$$

using the rotation method and (p,p)-boundedness of the Hardy-Littlewood maximal operator (Theorem 1.1.1 (c)) we easily obtain the following theorem.

Theorem 2.3.3 *Suppose that* Ω *satisfies* (2.2.1) *and* $\Omega \in L^1(\mathbb{S}^{n-1})$. *Then* M_Ω *is of type* (p,p) *for* $1 < p \le \infty$.

We have got L^p-boundedness $(1 < p < \infty)$ of a class of singular integral operators with odd rough kernel by the rotation method, where the kernel Ω is integrable on $\mathbb{S}^{n-1}$ and is an odd function, but without any smoothness. We know that any function Ω on $\mathbb{S}^{n-1}$ can always be decomposed into a sum of an odd function and an even one on $\mathbb{S}^{n-1}$. Thus, to deal with the boundedness of the corresponding singular integral operator with general kernel, we just need to consider the case that Ω is an even function on $\mathbb{S}^{n-1}$. Early in 1956, Calderón and Zygmund [CaZ3] considered the L^p-boundedness $(1 < p < \infty)$ of the corresponding singular integral operator T_Ω as $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ is an even function. They obtained the following result.

Theorem 2.3.4 *Suppose that* Ω *satisfies* (2.2.1), (2.2.2) *and* $\Omega(-x') = \Omega(x')$ *for every* $x' \in \mathbb{S}^{n-1}$. *If* $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$, *then* T_Ω *is of type* (p,p) *for* $1 < p < \infty$.

The basic idea of the proof is to turn the even kernel into the odd kernel by the Riesz transform, then using Theorem 2.3.1 we can deduce the conclusion. However, here we do not give the detailed proof of Theorem 2.3.4. Instead, we will discuss this problem in more general situation, and obtain the same conclusion, where Theorem 2.3.4 is just a natural consequence. In detail, since $L\log^+ L(\mathbb{S}^{n-1})$ is a proper subspace of $H^1(\mathbb{S}^{n-1})$ (under the condition (2.2.2)), where $H^1(\mathbb{S}^{n-1})$ is Hardy space on $\mathbb{S}^{n-1}$, we will show Theorem 2.3.4 holds whenever $\Omega \in H^1(\mathbb{S}^{n-1})$ is an even function. First we give the definition of $H^1(\mathbb{S}^{n-1})$. Suppose that $\alpha(x') \in L^1(\mathbb{S}^{n-1}), x' \in \mathbb{S}^{n-1}$. Set

$$P^+\alpha(x') = \sup_{0<t<1} \left| \int_{\mathbb{S}^{n-1}} P_{tx'}(y')\alpha(y')d\sigma(y') \right|,$$

where

$$P_{tx'}(y') = \frac{1-t^2}{|y'-tx'|^n}, \quad y' \in \mathbb{S}^{n-1}.$$

Now define

$$H^1(\mathbb{S}^{n-1}) = \{\alpha \in L^1(\mathbb{S}^{n-1}) : \|P^+\alpha\|_{L^1(\mathbb{S}^{n-1})} < \infty\},$$

and write

$$\|\alpha\|_{H^1(\mathbb{S}^{n-1})} = \|P^+\alpha\|_{L^1(\mathbb{S}^{n-1})}.$$

An important property of $H^1(\mathbb{S}^{n-1})$ is its atom decomposition. We first give the definition of atom. We call a function a on $\mathbb{S}^{n-1}$ regular atom if $a \in L^\infty(\mathbb{S}^{n-1})$ and there exist $\xi \in \mathbb{S}^{n-1}$ and $0 < r \le 2$ such that

(i) $\operatorname{supp}(a) \subset \mathbb{S}^{n-1} \bigcap B(\xi, r)$, where $B(\xi, r) = \{y \in \mathbb{R}^n : |y-\xi| < r\}$;

(ii) $\|a\|_{L^\infty(\mathbb{S}^{n-1})} \le r^{-n+1}$;

(iii) $\displaystyle\int_{\mathbb{S}^{n-1}} a(x')d\sigma(x') = 0$.

A function a on $\mathbb{S}^{n-1}$ is called exceptional atom, if $a \in L^\infty(\mathbb{S}^{n-1})$ and $\|a\|_{L^\infty(\mathbb{S}^{n-1})} \le 1$.

Lemma 2.3.1 *For any $\alpha \in H^1(\mathbb{S}^{n-1})$, there exist complex number λ_j and atom a_j (regular or exceptional) such that $\alpha = \sum_j \lambda_j a_j$, and*

$$\|\alpha\|_{H^1(\mathbb{S}^{n-1})} \sim \sum_j |\lambda_j|.$$

Now we formulate main result.

Theorem 2.3.5 *Suppose that Ω satisfies* (2.2.1), (2.2.2), *and $\Omega(-x') = \Omega(x')$, for every $x' \in \mathbb{S}^{n-1}$. If $\Omega \in H^1(\mathbb{S}^{n-1})$, then T_Ω is of type (p,p) for $1 < p < \infty$.*

We prove the following lemma first.

Lemma 2.3.2 *Suppose that Ω satisfies (2.2.1), (2.2.2) and $\Omega \in H^1(\mathbb{S}^{n-1})$. Then*

$$\frac{\Omega(x)}{|x|^s}\chi_{\{\frac{1}{2}<|x|\le 2\}}(x) \in H^1(\mathbb{R}^n)$$

for every $s \in \mathbb{R}$. Here we denote by $H^1(\mathbb{R}^n)$ the real Hardy space on $\mathbb{R}^n$.

Proof. By Lemma 2.3.1 and

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0,$$

we have

$$\Omega(x') = \sum_j \lambda_j a_j(x'), \tag{2.3.2}$$

where $\{a_j\}$ is a regular atom sequence in $H^1(\mathbb{S}^{n-1})$. Assume, without loss of generality, that

$$\operatorname{supp}(a_j) \subset B(\xi_j', \rho_j)\bigcap \mathbb{S}^{n-1},$$

where $\xi_j' \in \mathbb{S}^{n-1}$.

First recall the definition of atom on real Hardy space $H^1(\mathbb{R}^n)$. The function b on $\mathbb{R}^n$ is called $H^1(\mathbb{R}^n)$ atom, if

(i) there exist $x_0 \in \mathbb{R}^n$ and $r > 0$ such that $\operatorname{supp}(b) \subset B(x_0, r)$;

(ii) $\displaystyle\int_{\mathbb{R}^n} b(x)dx = 0$;

(iii) $\|b\|_{L^\infty} \le r^{-n}$.

Moreover,

$$f \in H^1(\mathbb{R}^n) \Longleftrightarrow f = \sum_{j=1}^{\infty} \mu_j b_j \tag{2.3.3}$$

where b_j is an atom in $H^1(\mathbb{R}^n)$, and

$$\|f\|_{H^1} \sim \sum_{j=1}^{\infty} |\mu_j| < \infty.$$

For the above conclusion and other properties on real Hardy spaces, readers can refer to Lu [Lu1].

Now turn to the proof of Lemma 2.3.2. First extend a_j to $\mathbb{R}^n\backslash\{0\}$ along the radial direction, i.e., for $x \in \mathbb{R}^n \setminus \{0\}$, put

$$\widetilde{a_j}(x) = a_j\left(\frac{x}{|x|}\right).$$

We split it into two cases.

Case I $\rho_j \geqslant \frac{1}{4}$. Applying the assumption that a is a regular atom, we have

$$\frac{\widetilde{a}_j(x)}{|x|^s}\chi_{\{\frac{1}{2}<|x|\leq 2\}}(x) = Cb_j(x), \tag{2.3.4}$$

where b_j is an atom in $H^1(\mathbb{R}^n)$, and C is a constant depending only on n, s.

Case II $0 < \rho_j < \frac{1}{4}$. Set

$$E_k = \left\{x \in \mathbb{R}^n : \frac{1}{2} + 2(k-1)\rho_j < |x| \leq \frac{1}{2} + 2k\rho_j\right\}, \quad k = 1, 2, \cdots, \left[\frac{3}{4\rho_j}\right]$$

and

$$E_0 = \left\{x \in \mathbb{R}^n : \frac{1}{2} + \left[\frac{3}{4\rho_j}\right] 2\rho_j < |x| \leq 2\right\},$$

where $[t]$ denotes the largest integer not exceeding t (E_0 may be an empty set.) In this way, we have

$$\frac{\widetilde{a}_j(x)}{|x|^s}\chi_{\{\frac{1}{2}<|x|\leq 2\}}(x) = \sum_{k=0}^{[\frac{3}{4\rho_j}]} \frac{\widetilde{a}_j(x)}{|x|^s}\chi_{E_k}(x). \tag{2.3.5}$$

Applying the condition of a_j being a regular atom together with the construction of E_k, it is easy to see that, there exists a constant C depending only on n and s, but independent of K and ρ_j, such that

$$\frac{\widetilde{a}_j(x)}{|x|^s}\chi_{E_k}(x) = C\rho_j b_{jk}(x), \qquad k = 0, 1, \cdots, \left[\frac{3}{4\rho_j}\right],$$

where b_{jk} is an atom in $H^1(\mathbb{R}^n)$. Thus, by (2.3.2), (2.3.4) and (2.3.5) we conclude that

$$\frac{\Omega(x)}{|x|^s}\chi_{\{\frac{1}{2}<|x|\leq 2\}}(x)$$

can be expressed as the sum of a sequence of atoms on $H^1(\mathbb{R}^n)$, the sum of whose coefficients is dominated by $C\sum_{j=1}^{\infty}|\lambda_j| < \infty$. By (2.3.3) we have

$$\frac{\Omega(x)}{|x|^s}\chi_{\{\frac{1}{2}<|x|\leq 2\}}(x) \in H^1(\mathbb{R}^n)$$

and

$$\left\|\frac{\Omega(x)}{|x|^s}\chi_{\{\frac{1}{2}<|x|\leq 2\}}(x)\right\|_{H^1(\mathbb{R}^n)} \leq C\|\Omega\|_{H^1(\mathbb{S}^{n-1})}. \tag{2.3.6}$$

∎

Lemma 2.3.3 *Suppose that Ω satisfies the condition of Theorem 2.3.5 and R_j are the Riesz transform ($j = 1, 2, \cdots, n$). Then*

$$K_j(x) = R_j\left(\frac{\Omega(\cdot)}{|\cdot|^n}\right)(x) = \text{p.v.}C_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n} dy,$$

with $C_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$, which satisfies the following properties:

(i) $K_j(x)$ is a homogeneous function of order $-n$ on $\mathbb{R}^n$;

(ii) $K_j(-x) = -K_j(x)$, $\quad x \in \mathbb{R}^n$;

(iii) $\displaystyle\int_{\mathbb{S}^{n-1}} \left|K_j(x')\right| d\sigma(x') \le C\|\Omega\|_{H^1(\mathbb{S}^{n-1})}$.

Proof. Properties (i) and (ii) are obvious, so it suffices to check (iii). For $\frac{3}{4} < |x| < \frac{3}{2}$, set

$$\begin{aligned} K_j(x) = \text{p.v.}C_n &\int_{|y|\le\frac{1}{2}} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n} dy + C_n \int_{\frac{1}{2}<|y|\le 2} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n} dy \\ &+ C_n \int_{|y|>2} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n} dy \\ &:= C_n(I_1 + I_2 + I_3). \end{aligned}$$

Since Ω satisfies (2.2.2), it implies that

$$\begin{aligned} |I_1| &= \left| \int_{|y|\le\frac{1}{2}} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) \frac{\Omega(y)}{|y|^n} dy \right| \\ &= \left| \int_0^{\frac{1}{2}} \int_{\mathbb{S}^{n-1}} y' \cdot \nabla \left(\frac{x_j}{|x|^{n+1}} \right) (x - ry' t_{x,ry'}) \Omega(y') d\sigma(y') dr \right| \\ &\le \frac{1}{2} \|\Omega\|_{L^1(\mathbb{S}^{n-1})} \cdot \max_{\frac{1}{4}\le|x|\le\frac{7}{4}} \left| \nabla \left(\frac{x_j}{|x|^{n+1}} \right) \right| \\ &= C\|\Omega\|_{L^1(\mathbb{S}^{n-1})}, \end{aligned}$$

where $t_{x,ry'} \in [0,1]$.

Note that the Riesz transform is (H^1, L^1) bounded, therefore by Lemma 2.3.2 and (2.3.6) we have

$$\int_{3/4<|x|<3/2} |I_2| dx \le C \left\| \frac{\Omega(\cdot)}{|\cdot|^n} \chi_{\{\frac{1}{2}<|\cdot|\le 2\}}(\cdot) \right\|_{H^1(\mathbb{R}^n)} \le C\|\Omega\|_{H^1(\mathbb{S}^{n-1})}.$$

As to I_3, it follows that

$$\begin{aligned}
|I_3| &\le \int_{|y|>2} \frac{1}{|x-y|^n} \frac{|\Omega(y)|}{|y|^n} dy \\
&\le \int_{|y|>2} \frac{4^n}{|y|^{2n}} |\Omega(y)| dy \\
&= C\|\Omega\|_{L^1(\mathbb{S}^{n-1})}.
\end{aligned}$$

Thus, we have

$$K_j(x)\chi_{\{\frac{3}{4}<|x|<\frac{3}{2}\}}(x) \in L^1(\mathbb{R}^n).$$

This fact is equivalent to $K_j(x') \in L^1(\mathbb{S}^{n-1})$ and

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} |K_j(x')| d\sigma(x') &= C \int_{\frac{3}{4}<|x|<\frac{3}{2}} |K_j(x)| dx \\
&\le C\|\Omega\|_{L^1(\mathbb{S}^{n-1})} + C\|\Omega\|_{H^1(\mathbb{S}^{n-1})} \\
&\le C\|\Omega\|_{H^1(\mathbb{S}^{n-1})}.
\end{aligned}$$

■

The Proof of Theorem 2.3.5. Let us first illustrate the following fact that

$$\sum_{j=1}^n R_j^2 = -I, \tag{2.3.7}$$

where I is an identity operator. In fact, for any Schwartz function f, from (2.0.3) it follows that

$$\left(\widehat{\sum_{j=1}^n R_j^2 f}\right)(\xi) = \sum_{j=1}^n \left(-i\frac{\xi_j}{|\xi|}\right)^2 \hat{f}(\xi) = -\hat{f}(\xi).$$

Then (2.3.7) follows.

By (2.3.7), we have

$$T_\Omega = \sum_{j=1}^n -R_j^2 T_\Omega := -\sum_{j=1}^n R_j T_j, \tag{2.3.8}$$

where $T_j = R_j T_\Omega$. If we denote the kernel of T_j by K_j, then

$$K_j(x) = R_j\left(\frac{\Omega(\cdot)}{|\cdot|^n}\right)(x), \qquad j = 1, 2, \cdots, n.$$

By Lemma 2.3.3, let $K_j(x) = \dfrac{V_j(x)}{|x|^n}$, where V_j satisfies (2.2.1) and is an odd integrable function on $\mathbb{S}^{n-1}$. Thus, by Theorem 2.3.1, T_j can be extended to be a linear bounded operator on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. Combining it with L^p boundedness of R_j (Theorem 2.1.4) and (2.3.8), we deduce that T_Ω is bounded on $L^p(\mathbb{R}^n)(1 < p < \infty)$.

Next we give the L^p boundedness $(1 < p < \infty)$ of the corresponding maximal singular integral operator T^*_Ω for $\Omega \in H^1(\mathbb{S}^{n-1})$.

Theorem 2.3.6 *Suppose that Ω satisfies the conditions in Theorem 2.3.5. Then the maximal operator T^*_Ω,*

$$T^*_\Omega f(x) = \sup_{\varepsilon>0} \left| \int_{|y|\geq\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy \right|,$$

is of type (p,p) $(1 < p < \infty)$.

Proof. Choose a radial function $\varphi \in C^\infty$ such that $0 \leq \varphi \leq 1$ and $\varphi(x) = 0$ when $|x| \leq \frac{1}{4}$ as well as $\varphi(x) = 1$ when $|x| \geq \frac{1}{2}$. Then

$$\begin{aligned} T^\varepsilon_\Omega f(x) :=& \int_{|x-y|\geq\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \\ =& \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \varphi\left(\frac{x-y}{\varepsilon}\right) f(y)dy \\ &- \int_{|x-y|<\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} \varphi\left(\frac{x-y}{\varepsilon}\right) f(y)dy. \end{aligned}$$

It follows from (2.3.1) that

$$\begin{aligned} &\sup_{\varepsilon>0} \left| \int_{|x-y|<\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} \varphi\left(\frac{x-y}{\varepsilon}\right) f(y)dy \right| \\ &\leq \sup_{\varepsilon>0} \int_{\varepsilon/4<|x-y|<\varepsilon} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy \leq M_\Omega f(x). \end{aligned}$$

Theorem 2.3.3 immediately implies that operator:

$$f \to \int_{|x-y|<\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} \varphi\left(\frac{x-y}{\varepsilon}\right) f(y)dy$$

is bounded on L^p $(1 < p \le \infty)$. Thus it suffices to show the smooth truncated maximal singular integral operator

$$\widetilde{T}^*_\Omega f(x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \varphi\left(\frac{x-y}{\varepsilon}\right) f(y)dy \right|$$

is bounded on L^p $(1 < p < \infty)$. For $1 \le j \le n$, set

$$K_j(x) = R_j\left(\frac{\Omega(\cdot)}{|\cdot|^n}\right)(x)$$

and

$$\widetilde{V}_j(x) = R_j\left(\frac{\Omega(\cdot)\varphi(\cdot)}{|\cdot|^n}\right)(x).$$

Put

$$\widetilde{T}^\varepsilon_\Omega f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \varphi\left(\frac{x-y}{\varepsilon}\right) f(y)dy.$$

Then

$$\widetilde{T}^\varepsilon_\Omega f = -\sum_{j=1}^n R_j\left(\frac{\Omega(\cdot)}{|\cdot|^n}\varphi\left(\frac{\cdot}{\varepsilon}\right)\right) * R_j(f) = -\sum_{j=1}^n \frac{1}{\varepsilon^n}\widetilde{V}_j\left(\frac{\cdot}{\varepsilon}\right) * R_j(f). \quad (2.3.9)$$

We need the following Lemma 2.3.4 whose proof will be given after that of Theorem 2.3.6.

Lemma 2.3.4 *There exists a function ω_j on $\mathbb{R}^n$ satisfying* (2.2.1), *and $\omega_j(x') \in L^1(\mathbb{S}^{n-1})$ such that*

(i) $|\widetilde{V}_j(x)| \le \omega_j(x), \quad |x| \le 1;$

(ii) $|\widetilde{V}_j(x) - K_j(x)| \le C\|\omega\|_{L^1(\mathbb{S}^{n-1})}|x|^{-n-1}, \quad |x| > 1.$

The above lemma and (2.3.9) implies that

$$\begin{aligned}\left|\widetilde{T}^\varepsilon_\Omega f(x)\right| &= \left|\sum_{j=1}^n \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \widetilde{V}_j\left(\frac{x-y}{\varepsilon}\right)(R_j f)(y)dy\right| \\ &\le I_1(f,\varepsilon) + I_2(f,\varepsilon) + I_3(f,\varepsilon),\end{aligned}$$

where

$$I_1(f,\varepsilon) = \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \left| \int_{|x-y|>\varepsilon} K_j\left(\frac{x-y}{\varepsilon}\right)(R_j f)(y)dy \right|,$$

$$I_2(f,\varepsilon) = \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \left| \int_{|x-y|>\varepsilon} \left[\widetilde{V}_j\left(\frac{x-y}{\varepsilon}\right) - K_j\left(\frac{x-y}{\varepsilon}\right)\right](R_j f)(y)dy \right|,$$

$$I_3(f,\varepsilon) = \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \left| \int_{|x-y|<\varepsilon} \widetilde{V}_j\left(\frac{x-y}{\varepsilon}\right)(R_j f)(y)dy \right|.$$

Since

$$\sup_{\varepsilon>0} |I_1(f,\varepsilon)| \le \sum_{j=1}^{n} S_j^*(R_j f)(x),$$

where S_j^* is the maximal singular integral operator with the kernel K_j for $1 \le j \le n$. By Lemma 2.3.3, Theorem 2.3.2 and L^p-boundedness of the Riesz transform, we can easily deduce that

$$\sup_{\varepsilon>0} I_1(f,\varepsilon)$$

is bounded on L^p $(1 < p < \infty)$. On the other hand, applying Lemma 2.3.4 (ii), we have

$$\begin{aligned}
\sup_{\varepsilon>0} I_2(f,\varepsilon) &\le \sup_{\varepsilon>0} C \cdot \sum_{j=1}^{n} \varepsilon \int_{|x-y|>\varepsilon} \frac{|R_j f(y)|}{|x-y|^{n+1}} dy \\
&\le C \cdot \sum_{j=1}^{n} M(R_j f)(x),
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator. The L^p-boundedness of M (Theorem 1.1.1) and Theorem 2.1.4 imply that

$$\sup_{\varepsilon>0} I_2(f,\varepsilon)$$

is also bounded on L^p $(1 < p < \infty)$. Finally, it follows from Lemma 2.3.4 (i) that

$$\begin{aligned}
\sup_{\varepsilon>0} I_3(f,\varepsilon) &\le \sup_{\varepsilon>0} \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \int_{|x-y|<\varepsilon} |\omega_j(x-y)||R_j f(y)|dy \\
&\le \sum_{j=1}^{n} M_{\omega_j}(R_j f)(x).
\end{aligned}$$

Thus we obtain that

$$\sup_{\varepsilon>0} I_3(f,\varepsilon)$$

is also bounded on L^p $(1<p<\infty)$ by applying Theorem 2.3.3 and Theorem 2.1.4. Therefore $\widetilde{T}^*_\Omega$ is bounded on L^p $(1<p<\infty)$. This completes the proof of Theorem 2.3.6. ∎

The proof of Lemma 2.3.4. Suppose $|x|>1$. Since $\varphi(y)=1$ when $|y|\geq\frac{1}{2}$, it follows from the vanishing condition (2.2.2) of Ω that

$$\begin{aligned}
\left|\widetilde{V}_j(x)-K_j(x)\right| &\leq \left|\int_{\mathbb{R}^n}\frac{x_j-y_j}{|x-y|^{n+1}}(\varphi(y)-1)\frac{\Omega(y)}{|y|^n}dy\right| \\
&\leq C\int_{|y|<\frac{1}{2}}\frac{|\Omega(y)|}{|y|^n}\left|\frac{x_j-y_j}{|x-y|^{n+1}}-\frac{x_j}{|x|^{n+1}}\right|dy \\
&\leq \frac{C}{|x|^{n+1}}\int_{|y|<\frac{1}{2}}\frac{|\Omega(y)|}{|y|^{n-1}}dy \\
&\leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1})}|x|^{-n-1}.
\end{aligned}$$

Now let $|x|\leq 1$. Note that $|\widetilde{V}_j(x)|\leq C\|\Omega\|_{L^1}$ when $|x|\leq\frac{1}{8}$, since $\varphi(y)\equiv 0$ when $|y|\leq\frac{1}{4}$. When $\frac{1}{8}\leq|x|\leq 1$, we have that

$$\begin{aligned}
&\left|\widetilde{V}_j(x)-\varphi(x)K_j(x)\right| \\
&\leq \int_{|y|>2}\frac{|\Omega(y)|}{|y|^n}\,|\varphi(y)-\varphi(x)|\,\frac{|x_j-y_j|}{|x-y|^{n+1}}dy \\
&\quad+\int_{\frac{1}{16}<|y|\leq 2}\frac{|\Omega(y)|}{|y|^n}|\varphi(y)-\varphi(x)|\frac{|x_j-y_j|}{|x-y|^{n+1}}dy \\
&\quad+\int_{0<|y|\leq\frac{1}{16}}\frac{|\Omega(y)|}{|y|^n}|\varphi(y)-\varphi(x)|\left|\frac{x_j-y_j}{|x-y|^{n+1}}-\frac{x_j}{|x|^{n+1}}\right|dy \\
&:= P_1(x)+P_2(x)+P_3(x).
\end{aligned}$$

We next calculate the estimate of P_1, P_2 and P_3, respectively.

Note that if $\frac{1}{8}\leq|x|\leq 1$ and $|y|>2$, then $|x-y|\geq C|y|$. Thus

$$P_1(x)\leq C\int_{|y|>2}\frac{|\Omega(y)|}{|y|^{2n}}dy\leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1})}.$$

The smoothness of φ implies $|\varphi(x)-\varphi(y)|\leq C|x-y|$. So for $\frac{1}{8}\leq|x|\leq 1$, we have that

$$P_2(x) \le C \int_{\frac{1}{16}<|y|\le 2} \frac{|\Omega(y)|}{|y|^n} \frac{dy}{|x-y|^{n-1}}$$
$$\le C|x|^{n-\frac{3}{2}} \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\frac{1}{2}}|x-y|^{n-1}} dy.$$

As to P_3, we have that

$$P_3(x) \le C \int_{0<|y|\le \frac{1}{16}} \frac{|\Omega(y)|}{|y|^{n-1}} dy \le C\|\Omega\|_{L^1(\mathbb{S}^{n-1})}.$$

Let

$$\eta(x) = |x|^{n-\frac{3}{2}} \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\frac{1}{2}}|x-y|^{n-1}} dy$$

and

$$V_j(x) = |x|^n K_j(x).$$

Take $\omega_j(x) = C\left(|V_j(x)| + \|\Omega\|_{L^1(\mathbb{S}^{n-1})} + \eta(x)\right)$, then clearly ω_j is a homogeneous function of degree 0, i.e., satisfying (2.2.1). Moreover, $\left|\widetilde{V}_j(x)\right| \le \omega_j(x)$ when $|x| \le 1$. We will next prove that $\omega_j(x') \in L^1(\mathbb{S}^{n-1})$. In fact, we only need to show η is integrable on $\left\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\right\}$.

Obviously, if $|y| \le \frac{1}{4}$, then we have that

$$\int_{\frac{1}{2}<|x|<2} \frac{|x|^{n-\frac{3}{2}}}{|x-y|^{n-1}} dx \le C_0;$$

if $|y| \ge 4$, then

$$\int_{\frac{1}{2}<|x|<2} \frac{|x|^{n-\frac{3}{2}}}{|x-y|^{n-1}} dx \le C_1|y|^{-n+1};$$

and if $\frac{1}{4} < |y| < 4$, then

$$\int_{\frac{1}{2}<|x|<2} \frac{|x|^{n-\frac{3}{2}}}{|x-y|^{n-1}} dx \le C_2 \int_{|x-y|<6} \frac{dx}{|x-y|^{n-1}} \le C_2'.$$

It follows from the above estimates that

$$\int_{\frac{1}{2}<|x|<2} \eta(x) dx \le C\|\Omega\|_{L^1(\mathbb{S}^{n-1})}.$$

This finishes the proof of Lemma 2.3.4.

As a natural consequence of Theorem 2.3.6, we have

Theorem 2.3.7 *Suppose that* Ω *satisfies* (2.2.1) *and* (2.2.2). *If* $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$, *then the maximal operator* T_Ω^* *is of type* (p,p) $(1 < p < \infty)$.

Next we will give the weighted boundedness of singular integral operator T_Ω with rough kernel and the corresponding maximal operator T_Ω^*. In the proof of the weighted boundedness of the Calderón-Zygmund singular integral operator (Theorem 2.1.6), we used the technique of good-λ inequality. For singular integral operators with homogeneous kernel, in the proof of their weighted boundedness (Theorem 2.2.3) we essentially still use this inequality. Since the kernel Ω of singular integral operator with rough kernel does not have any smoothness on the unit sphere, the good-λ inequality is not applicable. In order to deal with this problem, Duoandikoetxea and Rubio de Francia synthetically used the Fourier transform estimate, weighted Littlewood-Paley theory and Stein-Weiss interpolation method with change of measure, then obtained the weighted boundedness of T_Ω and T_Ω^*. In their proof, the weighted boundedness of the maximal operator M_Ω with rough kernel (for its definition, see (2.3.1)) is needed, while the latter itself is of great significance.

Theorem 2.3.8 *Suppose that* Ω *is a homogeneous function of degree 0 on* $\mathbb{R}^n$ *and* $\Omega \in L^q(\mathbb{S}^{n-1})$, $q > 1$. *If* p, q *and the weight function* ω *satisfy one of the following statements:*

(i) $q' \le p < \infty$, $p \neq 1$ *and* $\omega \in A_{p/q'}$;

(ii) $1 < p \le q$, $p \neq \infty$ *and* $\omega^{1-p'} \in A_{p'/q'}$;

(iii) $1 < p < \infty$, *and* $\omega^{q'} \in A_p$,

then M_Ω *is bounded on* $L^p(\omega)$.

Proof. It suffices to prove this result under the statements (i) and (ii). The statement (iii) will be easily deduced by applying a method similar to that in the proof of Theorem 2.2.4.

The conclusion under the statement (i) is easy to prove. In fact,

$$M_\Omega f(x) \le \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left[M\left(|f|^{q'}\right)(x)\right]^{1/q'},$$

where M is the Hardy-Littlewood maximal operator. If $p/q' > 1$, then the conclusion follows directly from Theorem 1.4.3. If $p/q' = 1$, the conclusion can be obtained by applying the method in the proof of Theorem 2.2.3.

Now consider the case of the statement (ii). Let us first introduce some notations. For $j \in \mathbb{Z}$, set

$$K_{\Omega,j}(x) = \Omega(x)\chi_{[2^j,2^{j+1})}(x)|x|^{-n},$$

$$T_{\Omega,j}f(x) = K_{\Omega,j} * f(x) = \int_{2^j \le |x-y| < 2^{j+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$$

and

$$g_\Omega(f)(x) = \left(\sum_{j\in\mathbb{Z}} |T_{\Omega,j}f(x)|^2 \right)^{\frac{1}{2}}. \tag{2.3.10}$$

It is clear that there exists $C > 0$, such that

$$M_\Omega f(x) \le C \cdot \sup_j T_{|\Omega|,j}(|f|)(x). \tag{2.3.11}$$

Let

$$\Omega_0(x) = |\Omega(x)| - \frac{\|\Omega\|_{L^1(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|},$$

then $\Omega_0 \in L^q(\mathbb{S}^{n-1})$ satisfies (2.2.1) and (2.2.2). By (2.3.11) we have

$$\begin{aligned} M_\Omega f(x) \quad &\le C \sup_j \Bigg(\int_{2^j \le |x-y| < 2^{j+1}} \frac{\Omega_0(x-y)}{|x-y|^n} f(y)dy \\ &\qquad + \frac{\|\Omega\|_{L^1}}{|\mathbb{S}^{n-1}|} \int_{2^j \le |x-y| < 2^{j+1}} \frac{|f(y)|}{|x-y|^n} dy \Bigg) \qquad (2.3.12) \\ &\le C g_{\Omega_0}(|f|)(x) + CMf(x). \end{aligned}$$

We split the proof of this theorem into that of several propositions by the value of q. Analogous to the statement (i), we merely need to consider the case $1 < p < q$.

Proposition 2.3.1 *Suppose that* Ω *satisfies* (2.2.1) *and* $\Omega \in L^q(\mathbb{S}^{n-1})$, $q > \max\{p, 2\}$. *If* $\omega^{1-p'} \in A_{p'/q'}$, *then there exists a constant* $C > 0$ *such that*

$$\|M_\Omega f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

for every $f \in L^p(\omega)$.

It follows from $\omega^{1-p'} \in A_{p'/q'} \subset A_{p'}$ and the property (I) (Proposition 1.4.1) of A_p that $\omega \in A_p$. Hence there exists a constant $C > 0$ such that

$$\|Mf\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

by applying Theorem 1.4.3. Noting (2.3.12), it remains to check

$$\|g_{\Omega_0}(|f|)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)} \tag{2.3.13}$$

under the hypothesis of Proposition 2.3.1, where

$$g_{\Omega_0}(|f|)(x) = \left(\sum_j |T_{\Omega_0,j}(|f|)(x)|^2\right)^{\frac{1}{2}}.$$

For any sequence $\varepsilon = \{\varepsilon_j\},\ \varepsilon_j = \pm 1$, define a linear operator

$$T_{\varepsilon,\Omega_0} f(x) = \sum_{j\in\mathbb{Z}} \varepsilon_j (K_{\Omega_0,j} * f)(x).$$

Then by a result of Rademacher function (see Kurtz [Ku]), to prove (2.3.13), it suffices to show that there exist C, independent of f and $\{\varepsilon_j\}$, such that

$$\|T_{\varepsilon,\Omega_0} f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}. \tag{2.3.14}$$

Thus we reduce the proof of Proposition 2.3.1 to the validation of (2.3.14). Next we will show that (2.3.14) holds by two lemmas.

Lemma 2.3.5 *Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfies* (2.2.1) *and* (2.2.2). *If $p > q',\ q > 2$ and $\omega \in A_{p/q'}$, then there exists a constant $C > 0$, independent of f and $\{\varepsilon_j\}$, such that*

$$\|T_{\varepsilon,\Omega} f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Proof. Take a radial function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \le \psi \le 1, \quad \text{supp}\psi \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} \le |x| \le 2\right\}$$

and

$$\sum_{k\in\mathbb{Z}} \psi^2(2^k x) = 1, \quad x \ne 0.$$

For any $f \in \mathscr{S}(\mathbb{R}^n)$, define a multiplier

$$S_k : \widehat{(S_k f)}(\xi) = \psi(2^k\xi)\hat{f}(\xi).$$

Then

$$\sum_{k\in\mathbb{Z}} S_k^2 f(x) = f(x).$$

Consequently, it follows that

$$\begin{aligned} T_{\varepsilon,\Omega} f(x) &= \sum_{j\in\mathbb{Z}} \varepsilon_j \left(K_{\Omega,j} * f\right)(x) \\ &= \sum_{j\in\mathbb{Z}} \varepsilon_j K_{\Omega,j} * \left(\sum_{k\in\mathbb{Z}} (S_{j+k}^2 f\right)(x) \\ &= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \varepsilon_j S_{j+k} \left(K_{\Omega,j} * S_{j+k} f\right)(x) \\ &:= \sum_{k\in\mathbb{Z}} T_{\varepsilon,\Omega}^k f(x). \end{aligned} \tag{2.3.15}$$

By the Plancherel theorem, we have

$$\begin{aligned} \left\| T_{\varepsilon,\Omega}^k f \right\|_{L^2}^2 &\le C \|\{\varepsilon_j\}\|_{l^\infty} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |S_{j+k}(K_{\Omega,j} * S_{j+k} f)(x)|^2 \, dx \\ &\le C \sum_{j\in\mathbb{Z}} \int_{2^{-j-k-1}\le|\xi|\le 2^{-j-k+1}} \left| \widehat{K}_{\Omega,j}(\xi) \right|^2 \left| \widehat{f}(\xi) \right|^2 d\xi. \end{aligned}$$

Next we will show that there exist $C > 0$ and $0 < \theta < 1$, such that

$$|\widehat{K}_{\Omega,j}(\xi)| \le C \min\left\{ |2^j\xi|^\theta,\ |2^j\xi|^{-\theta} \right\} \tag{2.3.16}$$

for every $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. Obviously, there exists C, depending only on Ω and n, such that

$$|\widehat{K}_{\Omega,j}(\xi)| \le C. \tag{2.3.17}$$

Since Ω satisfies the vanishing condition (2.2.2), it follows that

$$\left| \widehat{K}_{\Omega,j}(\xi) \right| = \left| \int_{2^j\le|x|\le 2^{j+1}} \frac{\Omega(x)}{|x|^n} \left(e^{-2\pi i x\cdot\xi} - 1 \right) dx \right| \le C|2^j\xi|.$$

This formula and (2.3.17) imply

$$\left| \widehat{K}_{\Omega,j}(\xi) \right| \le C|2^j\xi|^\theta$$

for any $0 < \theta < 1$, where C is independent of j and ξ.

On the other hand, Hölder's inequality implies

$$\left|\widehat{K}_{\Omega,j}(\xi)\right|^2 \leq \log 2 \cdot \int_{2^j}^{2^{j+1}} |I_r(\xi)|^2 \frac{dr}{r},$$

where

$$I_r(\xi) = \int_{\mathbb{S}^{n-1}} \Omega(x') e^{-2\pi i r x' \cdot \xi} d\sigma(x').$$

Since

$$\begin{aligned}\left|\int_{2^j}^{2^{j+1}} e^{-2\pi i r \xi \cdot (x'-y')} \frac{dr}{r}\right| &\leq C \min\left\{1, |2^j \xi \cdot (x'-y')|^{-1}\right\} \\ &\leq C|2^j\xi|^{-\alpha} \left|\xi' \cdot (x'-y')\right|^{-\alpha}\end{aligned}$$

with $0 < \alpha < 1$ and $aq' < 1$, Hölder's inequality again yields that

$$\begin{aligned}\left|\widehat{K}_{\Omega,j}(\xi)\right| &\leq C \left(\int_{2^j}^{2^{j+1}} \left|\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \Omega(x')\overline{\Omega(y')} e^{-2\pi i r\xi\cdot(x'-y')} d\sigma(x') d\sigma(y')\right| \frac{dr}{r}\right)^{1/2} \\ &\leq C \left|2^j \xi\right|^{-\alpha/2} \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left(\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \frac{d\sigma(x')d\sigma(y')}{|x'-y'|^{\alpha q'}}\right)^{1/2q'} \\ &\leq C \left|2^j\xi\right|^{-\alpha/2}.\end{aligned}$$

Choose $\theta = \alpha/2$. We then get (2.3.16) which concludes that

$$\left\|T^k_{\varepsilon,\Omega} f\right\|_{L^2} \leq C 2^{-\theta|k|} \|f\|_{L^2} \tag{2.3.18}$$

for every $k \in \mathbb{Z}$, where C is independent of $k, f, \{\varepsilon_j\}$ and $0 < \theta < 1$. Let us now show that there exists a constant $C > 0$, independent of f, $\{\varepsilon_j\}$ and $k \in \mathbb{Z}$, such that

$$\|T^k_{\varepsilon,\Omega} f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}. \tag{2.3.19}$$

Indeed, the conditions of Lemma 2.3.5 implies $\omega \in A_p$. By applying the weighted Littlewood-Paley theory (see Chapter 5, Theorem 5.2.3), there exists a constant C, independent of f and $\{\varepsilon_j\}$, such that

$$\begin{aligned}
\left\|T^k_{\varepsilon,\Omega}f\right\|_{L^p(\omega)} &= \left(\int_{\mathbb{R}^n}\left|\sum_{j\in\mathbb{Z}}\varepsilon_j S_{j+k}(K_{\Omega,j}*S_{j+k}f)(x)\right|^p\omega(x)dx\right)^{1/p}\\
&\le C\|\{\varepsilon_j\}\|_{l^\infty}\left\|\left(\sum_{j\in\mathbb{Z}}|S_{j+k}(K_{\Omega,j}*S_{j+k}f)(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)}\\
&\le C\left\|\left(\sum_{j\in\mathbb{Z}}|(K_{\Omega,j}*S_{j+k}f)(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)}.
\end{aligned}$$

For any fixed k, denote $h_j(x)=S_{j+k}f(x)$. Then

$$\begin{aligned}
&|K_{\Omega,j}*S_{j+k}f(x)|\\
&\quad\le\left(\int_{2^j\le|x-y|\le 2^{j+1}}\frac{|\Omega(x-y)|^q}{|x-y|^n}dy\right)^{1/q}\left(\int_{2^j\le|x-y|\le 2^{j+1}}\frac{|h_j(y)|^{q'}}{|x-y|^n}dy\right)^{1/q'}\\
&\quad\le C\left[M\left(|h_j|^{q'}\right)(x)\right]^{1/q'}.
\end{aligned}$$

Thus

$$\left\|T^k_{\varepsilon,\Omega}f\right\|_{L^p(\omega)}\le C\cdot\left\|\left\|M\left(|h_j|^{q'}\right)(\cdot)\right\|_{l^{2/q'}}\right\|^{1/q'}_{L^{p/q'}(\omega)}.$$

Since $\omega\in A_{p/q'}$, by the weighted norm inequality of vector-valued maximal operator (see Andersen and John [AnJ]) we obtain that

$$\begin{aligned}
\left\|T^k_{\varepsilon,\Omega}f\right\|_{L^p(\omega)} &\le C\left\|\left\|h_j^{q'}(\cdot)\right\|_{l^{2/q'}}\right\|^{1/q'}_{L^{p/q'}(\omega)}\\
&= C\left\|\left(\sum_{j\in\mathbb{Z}}|S_{j+k}f(\cdot)|^2\right)^{1/2}\right\|_{L^p(\omega)}\\
&\le C\|f\|_{L^p(\omega)}.
\end{aligned}$$

Here we use again the weighted Littlewood-Paley theory(Theorem 5.2.3). Thus we obtain (2.3.19).

Next we shall use the Stein-Weiss interpolation theorem with change of measure (Lemma 2.2.5) to complete the proof of Lemma 2.3.5. We deal with it in three cases.

Case 1: $p > 2$. Since $\omega \in A_{p/q'}$, Property (II) of A_p(Proposition 1.4.2) tells us that there exists $\sigma > 0$ such that $\omega^{1+\sigma} \in A_{p/q'}$. Take p_1 such that

$$\frac{(p_1 - p)}{(p - 2)} = \sigma,$$

then $p_1 > p$ and $\omega^{1+\sigma} \in A_{p_1/q'}$. The proof of (2.3.19) implies that there exists a constant C_1, independent of f and k, such that

$$\left\| T^k_{\varepsilon,\Omega} f \right\|_{L^{p_1}(\omega^{1+\sigma})} \le C_1 \|f\|_{L^{p_1}(\omega^{1+\sigma})}. \tag{2.3.20}$$

Set

$$t = \frac{p_1}{(1+\sigma)p},$$

then $0 < t < 1$ and

$$\frac{1}{p} = \frac{1-t}{2} + \frac{t}{p_1}.$$

Applying Lemma 2.2.5 to (2.3.18) and (2.3.20) yields

$$\left\| T^k_{\varepsilon,\Omega} f \right\|_{L^p(\omega)} \le C_1 2^{-\theta\gamma|k|} \|f\|_{L^p(\omega)}, \tag{2.3.21}$$

where $C_1, \theta, \gamma > 0$ are independent of f and $k \in \mathbb{Z}$.

Case 2: $p < 2$. Since $\omega \in A_{p/q'}$, there exists $\delta > 0$ such that $\omega^{1+\delta} \in A_{p/q'}$. Again Property (II) of A_p implies that there exists $q' < l < p$ such that $\omega^{1+\delta} \in A_{l/q'}$. Thus we can choose σ, p_0 such that $0 < \sigma \le \delta$ and $q' < l \le p_0 < p$ satisfying

$$\sigma = \frac{(p - p_0)}{(2 - p)}, \quad \omega^{1+\sigma} \in A_{p_0/q'}.$$

Actually, if $\delta = (p-l)/(2-p)$, then we can take $\sigma = \delta$ and $p_0 = l$. If $\delta < (p-l)/(2-p)$, then we can take $\sigma = \delta$ and $l < p_0 < p$ such that $\sigma = (p-p_0)/(2-p)$. If $\delta > (p-l)/(2-p)$, then we can take $0 < \sigma < \delta$ and $p_0 = l$ such that $\sigma = (p-p_0)/(2-p)$. We still have $\omega^{1+\sigma} \in A_{p_0/q'}$. Therefore it follows from (2.3.19) that

$$\left\| T^k_{\varepsilon,\Omega} f \right\|_{L^{p_0}(\omega^{1+\sigma})} \le C_2 \|f\|_{L^{p_0}(\omega^{1+\sigma})}. \tag{2.3.22}$$

Analogously, set $t = p_0/(1+\sigma)p$, then applying Lemma 2.2.5 to (2.3.18) and (2.3.22) yields

$$\left\| T^k_{\varepsilon,\Omega} f \right\|_{L^p(\omega)} \le C_2 2^{-\theta\gamma'|k|} \|f\|_{L^p(\omega)}, \tag{2.3.23}$$

where $C_2, \theta, \gamma' > 0$ are independent of f and $k \in \mathbb{Z}$.

Case 3: $p = 2$. Since $\omega \in A_{2/q'}$, there exists $\sigma > 0$ such that $\omega^{1+\delta} \in A_{2/q'}$. Consequently, we have

$$\left\| T^k_{\varepsilon,\Omega} f \right\|_{L^2(\omega^{1+\sigma})} \leq C_3 \|f\|_{L^2(\omega^{1+\sigma})}. \tag{2.3.24}$$

Set $t = 1/(1+\sigma)$, then applying Lemma 2.2.5 to (2.3.18) and (2.3.24) leads to

$$\left\| T^k_{\varepsilon,\Omega} f \right\|_{L^2(\omega)} \leq C_3 2^{-\theta\gamma''|k|} \|f\|_{L^2(\omega)}, \tag{2.3.25}$$

where $C_3, \theta, \gamma'' > 0$ are independent of f and $k \in \mathbb{Z}$.

Now put $C = \max\{C_1, C_2, C_3\}$, $\eta = \min\{\gamma, \gamma', \gamma''\}$, then for $p > q'$ and $q > 2$, by (2.3.21), (2.3.23) and (2.3.25) we obtain

$$\left\| T^k_{\varepsilon,\Omega} f \right\|_{L^p(\omega)} \leq C 2^{-\theta\eta|k|} \|f\|_{L^p(\omega)}.$$

This formula together with (2.3.15) leads to Lemma 2.3.5. ■

Lemma 2.3.6 *Suppose that $\Omega(x') \in L^q(\mathbb{S}^{n-1})$ satisfies* (2.2.1) *and* (2.2.2)*. If $q > \max\{p, 2\}$ and $\omega^{1-p'} \in A_{p'/q'}$, then there exists a constant $C > 0$, independent of f and $\{\varepsilon_j\}$, such that*

$$\|T_{\varepsilon,\Omega} f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

Using Lemma 2.3.5 and the dual method (see the proof of Theorem 2.2.3) we immediate obtain Lemma 2.3.6.

Thus the inequality (2.3.14) can be deduced from Lemma 2.3.6 under the condition of Proposition 2.3.1. Hence we complete the proof of Proposition 2.3.1.

Proposition 2.3.2 *Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfies (2.2.1). If $q > \max\{p, \frac{4}{3}\}$ and $\omega^{1-p'} \in A_{p'/q'}$, then there exists a constant $C > 0$, independent of f, such that*

$$\|M_\Omega f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

Clearly, it is enough to consider the case

$$\max\left\{p,\ \frac{4}{3}\right\} < q \leq 2.$$

Also from the proof of Proposition 2.3.1 we observe that the proof of Proposition 2.3.2 lies in the following lemma.

Lemma 2.3.7 *Suppose that* $\Omega(x') \in L^q(\mathbb{S}^{n-1})$ *satisfies* (2.2.1) *and* (2.2.2). *If* $q' < p$, $4/3 < q \le 2$ *and* $\omega \in A_{p/q'}$, *then there exists a constant* $C > 0$, *independent of* f *and* $\{\varepsilon_j\}$, *such that*

$$\|T_{\varepsilon,\Omega} f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Proof. From the proof of Lemma 2.3.5, it is easy to see that if we can show, under the assumption of Lemma 2.3.7, there exists a constant $C > 0$ independent of f and $\{\varepsilon_j\}$ such that

$$\left\|T^k_{\varepsilon,\Omega} f\right\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \tag{2.3.26}$$

then we can obtain the conclusion of Lemma 2.3.7 by applying the Stein-Weiss interpolation theorem with change of measure (Lemma 2.2.5) to (2.3.18) and (2.3.26).

First suppose that $\frac{4}{3} < q < 2$. Then $2 < q' < p$. Set

$$\widetilde{K}_{\Omega,j}(x) = |K_{\Omega,j}(x)|^{2-q}.$$

Then it follows that

$$\begin{aligned}|K_{\Omega,j} * g(x)|^2 &\le \left(\int_{\mathbb{R}^n} |K_{\Omega,j}(x-y)|^q dy\right)\left(\int_{\mathbb{R}^n} |K_{\Omega,j}(x-y)|^{2-q}|g(y)|^2 dy\right)\\ &\le C2^{j(n-nq)}\widetilde{K}_{\Omega,j} * \left(|g|^2\right)(x)\end{aligned}$$

and

$$\begin{aligned}\widetilde{K}_{\Omega,j} * (|h|)(x) &\le \int_{2^j \le |x-y| < 2^{j+1}} \left(\frac{|\Omega(x-y)|}{|x-y|^n}\right)^{2-q} |h(y)|dy\\ &\le C2^{-jn(2-q)} \int_{|x-y|<2^{j+1}} |\Omega(x-y)|^{2-q}|h(y)|dy\\ &\le C2^{-j(n-nq)} M_{\Omega^{2-q}}(h)(x).\end{aligned} \tag{2.3.27}$$

Since $\omega \in A_p$, by the weighted Littlewood Paley theory (see Theorem 5.2.3) we conclude that

$$\begin{aligned}
\left\|T^k_{\varepsilon,\Omega} f\right\|^2_{L^p(\omega)} &= \left(\int_{\mathbb{R}^n} \left|\sum_{j\in\mathbb{Z}} \varepsilon_j S_{j+k}(K_{\Omega,j} * S_{j+k} f)(x)\right|^p \omega(x)dx\right)^{2/p} \\
&\leq C \left\|\{\varepsilon_j\}\right\|_{l^\infty} \left\|\left(\sum_{j\in\mathbb{Z}} |S_{j+k}(K_{\Omega,j} * S_{j+k} f)(\cdot)|^2\right)^{1/2}\right\|^2_{L^p(\omega)} \\
&\leq C \left\|\left(\sum_{j\in\mathbb{Z}} |(K_{\Omega,j} * S_{j+k} f)(\cdot)|^2\right)^{1/2}\right\|^2_{L^p(\omega)} \\
&\leq C \left\|\left(\sum_{j\in\mathbb{Z}} 2^{j(n-nq)} \widetilde{K}_{\Omega,j} * |S_{j+k} f|^2(\cdot)\right)^{1/2}\right\|^2_{L^p(\omega)} \\
&= C \sup_{n} \left|\int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} 2^{j(n-nq)} \widetilde{K}_{\Omega,j} * |S_{j+k} f|^2(x)\right) h(x)dx\right|,
\end{aligned}$$

where the supremum is taken over all functions h satisfying

$$\|h\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \leq 1.$$

By (2.3.27) we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} 2^{j(n-nq)} \widetilde{K}_{\Omega,j} * |S_{j+k} f|^2(x)\right) h(x)dx \\
&= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} 2^{j(n-nq)} |S_{j+k} f(x)|^2 \left(\widetilde{K}_{\Omega,j} * h\right)(x)dx \\
&\leq C \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |S_{j+k} f(x)|^2 M_{\Omega^{2-q}}(h)(x)dx \\
&\leq C \left(\int_{\mathbb{R}^n} \left(\sum_{j} |S_{j+k} f(x)|^2\right)^{p/2} \omega(x)dx\right)^{2/p} \\
&\qquad \times \left(\int_{\mathbb{R}^n} [M_{\Omega^{2-q}}(h)(x)]^{(p/2)'} \omega(x)^{1-(p/2)'} dx\right)^{1/(p/2)'}.
\end{aligned}$$

Since $4/3 < q < 2$, if we set $r = q/(2-q)$ (correspondingly, $r' = q'/2$), then $\Omega^{2-q} \in L^r(\mathbb{S}^{n-1})$ and $r > \max\left\{\left(\frac{p}{2}\right)', 2\right\}$ as well as $(p/2)/r' = p/q'$. Thus by Proposition 2.3.1 we obtain

$$\|M_{\Omega^{2-q}}(h)\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \le C\|h\|_{L^{(p/2)'}(\omega^{1-(p/2)'})}. \tag{2.3.28}$$

Using again the weighted Littlewood-Paley theory together with (2.3.28) yields that

$$\begin{aligned}
\left\|T^k_{\varepsilon,\Omega} f\right\|^2_{L^p(\omega)} &\le C \sup_h \|h\|_{L^{(\frac{p}{2})'}\left(\omega^{1-(\frac{p}{2})'}\right)} \left(\int_{\mathbb{R}^n} \left(\sum_j |S_{j+k}f(x)|^2\right)^{\frac{p}{2}} \omega(x)dx\right)^{\frac{2}{p}} \\
&\le C \left\|\left(\sum_j |S_{j+k}f(\cdot)|^2\right)^{1/2}\right\|^2_{L^p(\omega)} \\
&\le C\|f\|^2_{L^p(\omega)}.
\end{aligned}$$

Thus we obtain (2.3.26) for $\frac{4}{3} < q < 2$.

Now consider $q = 2$, then we have $2 = q' < p$ and $\omega \in A_{p/2}$. By the property (i) and (ii) of A_p (Proposition 1.4.1 and Proposition 1.4.2), we can choose $\sigma > 0$ such that

(i) $(2-\sigma)' < p$,

(ii) $\omega \in A_{p/(2-\sigma)'}$,

(iii) $\frac{4}{3} < (2-\sigma) < 2$,

(iv) $\Omega \in L^2(\mathbb{S}^{n-1}) \subset L^{2-\sigma}(\mathbb{S}^{n-1})$.

Consequently by the conclusion for $\frac{4}{3} < q < 2$, (2.3.26) still holds for $q = 2$. Therefore we have proved Lemma 2.3.7. ■

Applying Lemma 2.3.7 and the dual method, we can obtain the conclusion of Proposition 2.3.2.

By induction, if Proposition 2.3.2 holds for

$$q > \max\left\{p, \frac{2^{m-1}}{2^{m-1}-1}\right\}, \quad m \ge 2,$$

then it still holds for $q > \max\{p, 2^m/(2^m-1)\}$. That is, we have the following result.

Proposition 2.3.3 *Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfies (2.2.1). If $m \in \mathbb{N}$, $m \geq 2$, $q > \max\{p, 2^m/(2^m-1)\}$ and $\omega^{1-p'} \in A_{p'/q'}$, then there exists a constant $C > 0$, independent of f, such that*

$$\|M_\Omega f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Now we turn to the proof of Theorem 2.3.8. If $p \geq 2$, then Theorem 2.3.8 follows from Proposition 2.3.1. If $p < 2$, then there exists $m \in \mathbb{N}$, $m \geq 2$ such that

$$\frac{2^m}{2^m-1} \leq p < \frac{2^{m-1}}{2^{m-1}-1}.$$

So $q > p$ is equivalent to $q > \max\{p,\ 2^m/(2^m-1)\}$. Thus by Proposition 2.3.3 we prove that Theorem 2.3.8 holds under the condition (ii). ■

Theorem 2.3.9 *Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ $(q > 1)$ satisfies (2.2.1) and (2.2.2). If p, q and weight function ω satisfy one of the following conditions:*

(i) $q' \leq p < \infty,\ p \neq 1$ *and* $\omega \in A_{p/q'}$;

(ii) $1 < p \leq q,\ p \neq \infty$ *and* $\omega^{1-p'} \in A_{p'/q'}$;

(iii) $1 < p < \infty$ *and* $\omega^{q'} \in A_p$,

then T_Ω is bounded on $L^p(\omega)$.

Proof. In fact, the proof of Theorem 2.3.8 under the condition (ii) has implied that of Theorem 2.3.9. First we give the proof of Theorem 2.3.9 under the condition (i). It suffices to consider the case $p > q'$. We use the notations as in Theorem 2.3.8. It follows that

$$\begin{aligned} T_\Omega f(x) &= \sum_j K_{\Omega,j} * \left(\sum_k \left(S^2_{j+k} f\right)(x)\right) \\ &= \sum_k \sum_j S_{j+k}(K_{\Omega,j} * S_{j+k} f)(x) \\ &:= \sum_k T^k_\Omega f(x). \end{aligned}$$

By the Plancherel theorem, if we replace $T^k_{\varepsilon,\Omega}$ by T^k_Ω, then (2.3.18) still holds. That is, there exist constants $C,\ \theta > 0$ independent of f and $k \in \mathbb{Z}$, such that

$$\left\|T^k_\Omega f\right\|_{L^2} \leq C2^{-\theta|k|}\|f\|_{L^2}. \tag{2.3.29}$$

If we can show, under the condition (i), there exists a constant $C > 0$, independent of f and $k \in \mathbb{Z}$, such that

$$\left\|T_\Omega^k f\right\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \tag{2.3.30}$$

then we can get the conclusion under the condition (i) by applying the Stein-Weiss interpolation theorem with change of measure (Lemma (2.2.5)) to (2.3.29) and (2.3.30).

If $q > 2$, then (2.3.30) follows from (2.3.19). If $q < 2$ (then $p > q' > 2$), analogous to the proof of Lemma 2.3.7, we have that

$$\begin{aligned}\left\|T_\Omega^k f\right\|^2_{L^p(\omega)} \le & C\sup_h \left|\int_{\mathbb{R}^n}\left(\sum_{j\in\mathbb{Z}} 2^{j(n-nq)}\widetilde{K}_{\Omega,j} * |S_{j+k}f|^2(x)\right) h(x)dx\right| \\ \le & C\left\|\left(\sum_j |S_{j+k}f(\cdot)|^2\right)^{1/2}\right\|^2_{L^p(\omega)} \\ & \times \sup_h \left(\int_{\mathbb{R}^n}[M_{\Omega^{2-q}}h(x)]^{(\frac{p}{2})'}\omega(x)^{1-(p/2)'}dx\right)^{1/(p/2)'},\end{aligned}$$

where the supremum is taken over all functions h satisfying

$$\|h\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \le 1.$$

Since $q < 2$, it follows that $r = q/(2-q) > (p/2)'$, $(p/2)/r' = p/q'$ and $\Omega^{2-q} \in L^r(\mathbb{S}^{n-1})$. It is clear that r, $(p/2)'$ and $\omega^{1-(p/2)'}$ satisfy the condition (ii) of Theorem 2.3.8 under the condition (i). Thus (2.3.28) holds. In this way, for $q < 2$, the formula (2.3.30) can be deduced from (2.3.28) and the weighted Littlewood-Paley theory.

The way of dealing with the case $q = 2$ is completely the same as Lemma 2.3.7. Here we will not give details. Thus we have proved Theorem 2.3.9 under the condition (i).

Applying the conclusion under the condition (i) and dual method (see the proof of Theorem 2.2.3), we can obtain Theorem 2.3.9 under the condition (ii).

Finally, by the conclusion under the condition (i) and (ii), together with the method in the proof of Theorem 2.2.4 we can show that Theorem 2.3.9 still holds under the condition (iii). Thus we complete the proof of Theorem 2.3.9. ∎

Remark 2.3.1 *As space is limited, all results on boundedness of T_Ω and M_Ω are only involved in L^p-boundedness. In 1967, Calderón, Weiss and Zygmund [CaWZ] showed that if $\Omega \in L(log^+ L)^{1-\varepsilon}(\mathbb{S}^{n-1}), 0 < \varepsilon < 1$ and Ω satisfies (2.2.2), then there is an $L^2(\mathbb{R}^n)$ function f such that $T_\Omega f \notin L^2(\mathbb{R}^n)$. For the weak type (1,1) boundedness of T_Ω and M_Ω, we will mentioned several results here. In 1988, Christ [Chr] proved that M_Ω is of weak type (1,1) when $\Omega \in L^q(S^1)(q > 1)$. Hofmann [Ho] showed that T_Ω is of weak type (1,1) if $\Omega \in L^q(\mathbb{S}^1)(q > 1)$ and satisfies (2.2.2). And Christ and Rubio de Francia [ChrR] improved the result of Christ above and proved that M_Ω is of weak type (1,1) if $\Omega \in Llog^+ L(\mathbb{S}^{n-1})$ for all $n \geq 2$. In the same paper, Christ and Rubio de Francia [ChrR] proved that T_Ω is of weak type (1,1) if $\Omega \in Llog^+ L(\mathbb{S}^{n-1})$ for $n \leq 5$ and satisfies (2.2.2). In 1996, Seeger [Se] proved that T_Ω is of weak type (1,1) when $\Omega \in Llog^+ L(\mathbb{S}^{n-1})$ for all $n \geq 2$ and satisfies (2.2.2).*

Remark 2.3.2 *It is still an open problem whether T_Ω is of weak type (1,1) when $\Omega \in H^1(\mathbb{S}^{n-1})$ with the condition (2.2.2). It is also an open problem whether M_Ω is of weak type (1,1) if $\Omega \in L^1(\mathbb{S}^{n-1})$.*

2.4 Commutators of singular integral operators

In 1965, Calderón defined the Calderón commutator in studying the boundedness of the Cauchy integral on Lipschitz curves, and its definition is

$$C_{h,\varphi}(f)(x) = \text{p.v.} \int_{-\infty}^{\infty} h\left(\frac{\varphi(x) - \varphi(y)}{x - y}\right) \frac{f(y)}{x - y} dy,$$

where $h \in C^\infty(\mathbb{R})$, φ is a Lipschitz function on $\mathbb{R}$. It is clear that, if $h(t) = (1 + it)^{-1}$, then $C_{h,\varphi}(f)$ is the Cauchy integral along the curve $y = \varphi(x)$; if $h = 1$, then $C_{h,\varphi}(f)$ is Hilbert transform; if $h(t) = t^k$ (k is a natural number), then $C_{h,\varphi}(f)$ is commutator of degree k of the Hilbert transform about φ.

In 1976, Coifman, Rochberg and Weiss studied the L^p boundedness ($1 < p < \infty$) of commutator $[b, T_\Omega]$ generated by the Calderón-Zygmund singular integral operator T_Ω and a function b, where $[b, T_\Omega]$ is defined by

$$[b, T_\Omega](f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^n} [b(x) - b(y)] f(y) dy, \qquad (2.4.1)$$

where Ω satisfies the condition of homogeneity of degree zero (2.2.1) and the vanishing condition (2.2.2), moreover, $\Omega \in \mathrm{Lip1}(\mathbb{S}^{n-1})$ and b is a $\mathrm{BMO}(\mathbb{R}^n)$ function. Using the L^p boundedness of commutator $[b, T_\Omega]$, Coifman, Rochberg and Weiss successfully gave a decomposition of Hardy space $H^1(\mathbb{R}^n)$.

The commutator defined by (2.4.1) is called CRW-type commutator, and CRW-type commutator plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order.

In this section we will introduce some results on L^p boundedness and weighted L^p boundedness ($1 < p < \infty$) of CRW-type commutator.

First we introduce the definition of the space of BMO (Bounded Mean Oscillation) functions on $\mathbb{R}^n$. Suppose that f is a locally integrable function on $\mathbb{R}^n$, let

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx,$$

where the supreme is taken over all cubes $Q \subset \mathbb{R}^n$, and

$$f_Q = \frac{1}{|Q|} \int_Q f(y) dy.$$

Define

$$\mathrm{BMO}(\mathbb{R}^n) = \{f : \|f\|_{\mathrm{BMO}} < \infty\}.$$

If one regards two functions whose difference is a constant as one, then space BMO is a Banach space with respect to norm $\|\cdot\|_{\mathrm{BMO}}$.

We next formulate some remarks about $\mathrm{BMO}(\mathbb{R}^n)$ (we refer to Lu[Lu1]).

Remark 2.4.1

(1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$, such that for all $f \in BMO(\mathbb{R}^n)$ and $\alpha > 0$

$$|\{x \in Q : |f(x) - f_Q| > \alpha\}| \le C_1 |Q| e^{-C_2\alpha/\|f\|_{BMO}}, \forall Q \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|f\|_{\mathrm{BMO}} \sim \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{\frac{1}{p}} \tag{2.4.2}$$

for $1 < p < \infty$.

(3) By the definition of BMO *and the Sharp maximal function (see Section 2.1), if $f \in L_{loc}(\mathbb{R}^n)$, then*

$$f \in \mathrm{BMO}(\mathbb{R}^n) \Longleftrightarrow M^\sharp f \in L^\infty(\mathbb{R}^n). \tag{2.4.3}$$

(4) If $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, then $f(\cdot - h)$, the translation of f, satisfies that $f(\cdot - h) \in \mathrm{BMO}(\mathbb{R}^n)$, and

$$\|f(\cdot - h)\|_{\mathrm{BMO}} = \|f\|_{\mathrm{BMO}}. \tag{2.4.4}$$

(5) If $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, and $\lambda > 0$, then $f(\lambda x) \in \mathrm{BMO}(\mathbb{R}^n)$, and

$$\|f(\lambda \cdot)\|_{\mathrm{BMO}} = \|f\|_{\mathrm{BMO}}. \tag{2.4.5}$$

(6) If $f \in \mathrm{BMO}(\mathbb{R}^n)$, then

$$\|f\|_{\mathrm{BMO}} \sim \sup_Q \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(x) - c| dx. \tag{2.4.6}$$

Theorem 2.4.1 *Suppose that $\Omega \in Lip1(\mathbb{S}^{n-1})$ satisfies (2.2.1) and (2.2.2). T_Ω is a singular integral operator with the kernel Ω. Then the following two statements hold.*

(i) If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$;

(ii) Suppose $1 < p_0 < \infty$ and $b \in \cup_{q>1} L^q_{loc}(\mathbb{R}^n)$. If $[b, T_\Omega]$ is bounded on $L^{p_0}(\mathbb{R}^n)$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Let us first establish the following lemma.

Lemma 2.4.1 *Suppose $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $1 < p < \infty$. Then for any $1 < s < p$, there exists a constant C, independent of b and f, such that*

$$M^\sharp \left([b, T_\Omega] f\right)(x) \le C\|b\|_{\mathrm{BMO}} \left[(M(|T_\Omega f|^s)(x))^{\frac{1}{s}} + (M(|f|^s)(x))^{\frac{1}{s}}\right], \quad x \in \mathbb{R}^n, \tag{2.4.7}$$

where M is the Hardy-Littlewood maximal operator.

Proof. For $x \in \mathbb{R}^n$, let Q be any cube containing x. For $y \in Q$, set

$$\begin{aligned}[b, T_\Omega](f)(y) &= (b(y) - b_Q) T_\Omega f(y) - T_\Omega \left((b - b_Q) f \chi_{4\sqrt{n}Q}\right)(y) \\ &\quad - T_\Omega \left((b - b_Q) f \chi_{(4\sqrt{n}Q)^c}\right)(y) \\ &:= a_1(y) - a_2(y) - a_3(y).\end{aligned}$$

It follows from Hölder's inequality and (2.4.2) that

$$\begin{aligned}\frac{1}{|Q|}\int_Q |a_1(y)| dy &\le \left(\frac{1}{|Q|}\int_Q |b(y) - b_Q|^{s'} dy\right)^{\frac{1}{s'}} \left(\frac{1}{|Q|}\int_Q |T_\Omega f(y)|^s dy\right)^{\frac{1}{s}} \\ &\le C\|b\|_{\mathrm{BMO}} \Big[M\left(|T_\Omega f|^s\right)(x)\Big]^{\frac{1}{s}}.\end{aligned} \tag{2.4.8}$$

Take $1 < u,\ q < \infty$ such that $uq = s$. Thus by the L^q boundedness of T_Ω (see Theorem 2.1.2), we conclude that

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |a_2(y)|dy &\le \left(\frac{1}{|Q|}\int_Q \left|T_\Omega\left((b-b_Q)f\chi_{4\sqrt{n}Q}\right)(y)\right|^q dy\right)^{\frac{1}{q}}\\
&\le C\left(\frac{1}{|Q|}\int_{4\sqrt{n}Q} |b(y)-b_Q|^q|f(y)|^q dy\right)^{\frac{1}{q}}\\
&\le C\left(\frac{1}{|Q|}\int_{4\sqrt{n}Q} |b(y)-b_Q|^{qu'} dy\right)^{\frac{1}{qu'}} \qquad (2.4.9)\\
&\quad\times\left(\frac{1}{|Q|}\int_{4\sqrt{n}Q} |f(y)|^{qu} dy\right)^{\frac{1}{qu}}\\
&\le C\|b\|_{\mathrm{BMO}}\left[M(|f|^s)(x)\right]^{\frac{1}{s}}.
\end{aligned}$$

Now denote the center and side length of Q by x_0 and d, respectively. If $y \in Q, z \in (4\sqrt{n}Q)^c$, then $|z-y| \sim |z-x_0| \sim |z-x|$. Since $\Omega(x') \in \mathrm{Lip}1(\mathbb{S}^{n-1})$, it follows from (2.1.2) that

$$\begin{aligned}
\left|\frac{\Omega(y-z)}{|y-z|^n} - \frac{\Omega(x_0-z)}{|x_0-z|^n}\right| &\le |\Omega(y-z)|\left|\frac{1}{|y-z|^n} - \frac{1}{|x_0-z|^n}\right|\\
&\quad+\frac{1}{|x_0-z|^n}\left|\Omega\left(\frac{y-z}{|y-z|}\right) - \Omega\left(\frac{x_0-z}{|x_0-z|}\right)\right|\\
&\le \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\frac{cd}{|x_0-z|^{n+1}} + \frac{c}{|x_0-z|^n}\frac{2d}{|x_0-z|}\\
&\le \frac{cd}{|z-x_0|^{n+1}}. \qquad (2.4.10)
\end{aligned}$$

Here note $|z-y| \ge 2|y-x_0|$. Thus, it follows from (2.4.10) that

$$\begin{aligned}|a_3(y) - a_3(x_0)| &\leq \int_{\mathbb{R}^n \setminus 4\sqrt{n}Q} \left| \frac{\Omega(y-z)}{|y-z|^n} - \frac{\Omega(x_0-z)}{|x_0-z|^n} \right| |b(z) - b_Q||f(z)|dz \\ &\leq Cd \left(\int_{\mathbb{R}^n \setminus 4\sqrt{n}Q} \frac{|b(z)-b_Q|^{s'}}{|z-x_0|^{n+1}} dz \right)^{\frac{1}{s'}} \\ &\quad \times \left(\int_{\mathbb{R}^n \setminus 4\sqrt{n}Q} \frac{|f(z)|^s}{|z-x_0|^{n+1}} dz \right)^{\frac{1}{s}} \\ &\leq C\|b\|_{\mathrm{BMO}} \Big[M(|f|^s)(x) \Big]^{\frac{1}{s}}. \end{aligned} \tag{2.4.11}$$

Thus from (2.1.20) and (2.4.8), (2.4.9) and (2.4.11) it follows that (2.4.7) holds. ∎

To complete the proof of Theorem 2.4.1 (i), we still need to introduce a result of Fefferman and Stein. Its proof can be found in Fefferman and Stein [FeS2].

Lemma 2.4.2 (Fefferman-Stein) *Let $1 \leq p_0 < \infty$. Then for any $p_0 \leq p < \infty$, there exists a constant C, independent of f, such that*

$$\|Mf\|_p \leq C\|M^\sharp f\|_p \tag{2.4.12}$$

for any function f satisfying $Mf \in L^{p_0}(\mathbb{R}^n)$.

Actually the proof of Theorem 2.4.1 (i) is a direct consequence of Lemma 2.4.1, Lemma 2.4.2, Theorem 1.1.1 and Theorem 2.1.2.

Now we pay attention to the proof of Theorem 2.4.1 (ii).

Without loss of generality, we may assume

$$\Big\| [b, T_\Omega] \Big\|_{L^{p_0} \to L^{p_0}} = 1.$$

We wish to prove that there exists a constant $A = A(p_0, \Omega)$ such that

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \leq A. \tag{2.4.13}$$

By (2.4.4) and (2.4.5), we merely need to prove (2.4.13) in the case $Q = Q_1$, where Q_1 is a cube whose center is at the origin and side length $1/\sqrt{n}$ parallel to the coordinates. Moreover we notice that $[b - b_{Q_1}, T_\Omega] = [b, T_\Omega]$. Thus we may assume $b_{Q_1} = 0$.

Now let

$$\psi(x) = (\mathrm{sgn}(b) - c_0)\chi_{Q_1}(x),$$

where

$$c_0 = \frac{1}{|Q_1|}\int_{Q_1} \mathrm{sgn}(b(x))dx.$$

Then ψ satisfies the following properties:

$$\|\psi\|_\infty \le 2, \tag{2.4.14}$$

$$\mathrm{supp}(\psi) \subset Q_1, \tag{2.4.15}$$

$$\int_{Q_1} \psi(x)dx = 0, \tag{2.4.16}$$

$$\psi(x)b(x) \ge 0, \tag{2.4.17}$$

$$\frac{1}{|Q_1|}\int \psi(x)b(x)dx = \frac{1}{|Q_1|}\int_{Q_1} |b(x)|dx := B. \tag{2.4.18}$$

Since Ω satisfies (2.2.1) and (2.2.2), there exists $0 < A_1 < 1$, such that

$$\sigma\left(\left\{x' \in \mathbb{S}^{n-1} : \Omega(x') \ge 2A_1\right\}\right) > 0, \tag{2.4.19}$$

where σ is a measure on $\mathbb{S}^{n-1}$ induced by the Lesbesgue measure. Let

$$\Lambda = \left\{x' \in \mathbb{S}^{n-1} : \Omega(x') \ge 2A_1\right\}.$$

Then Λ is a closed set. It is easy to see that, for any $x' \in \Lambda$ and $y' \in \mathbb{S}^{n-1}$, when $|x' - y'| < A_1$, we have $\Omega(y') \ge A_1$.

Set

$$G = \left\{x \in \mathbb{R}^n : |x| > A_2 = \frac{2}{A_1} + 1 \quad \text{and} \quad x' \in \Lambda\right\}.$$

Then when $x \in G$ and $y \in Q_1$, we have

$$|x - y| > \frac{5}{6}|x|$$

and $|x| > 2|y|$. Thus it implies by (2.1.11) that

$$\left|\frac{x}{|x|} - \frac{x-y}{|x-y|}\right| \le 2\frac{|y|}{|x|} \le \frac{2}{|x|} < A_1.$$

Since $x' \in \Lambda$, we have $\Omega\left(\frac{x-y}{|x-y|}\right) \geq A_1$. Thus from (2.4.17) and (2.4.18), it follows that

$$\begin{aligned} |T_\Omega(b\psi)(x)| &= \left|\int_{Q_1} \frac{\Omega(x-y)}{|x-y|^n} b(y)\psi(y)dy\right| \\ &\geq A_3|x|^{-n}B, \end{aligned} \tag{2.4.20}$$

where A_3 depends on A_1 and n.

On the other hand, when $x \in G$ and $y \in Q_1$, it follows from the condition of Ω and (2.1.12) that

$$\left|\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n}\right| \leq A_4' \frac{1}{|x|^{n+1}}, \tag{2.4.21}$$

where A_4' only depends on n and Ω. Thus by (2.4.14), (2.4.16) and (2.4.21) we conclude that

$$\begin{aligned} |b(x)T_\Omega(\psi)(x)| &\leq |b(x)| \int_{Q_1} \left|\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n}\right| |\psi(y)|dy \\ &\leq A_4|b(x)||x|^{-(n+1)}, \end{aligned} \tag{2.4.22}$$

where A_4 depends only on n and Ω. Therefore when $x \in G$, from (2.4.20) and (2.4.22) it follows that

$$\begin{aligned} |[b, T_\Omega](\psi)(x)| &\geq |T_\Omega(b\psi)(x)| - |b(x)T_\Omega(\psi)(x)| \\ &\geq A_3|x|^{-n}B - A_4|b(x)||x|^{-(n+1)}. \end{aligned} \tag{2.4.23}$$

Let

$$F = \left\{x \in G : |b(x)| > \left(\frac{BA_3}{2A_4}\right)|x| \ \text{ and } \ |x| < B^{\frac{p_0'}{n}}\right\}.$$

It follows from (2.4.23) that

$$\begin{aligned}
\|\psi\|_{p_0}^{p_0} &\geq \int_{\mathbb{R}^n} |[b, T_\Omega](\psi)(x)|^{p_0} dx \\
&\geq \int_{(G\setminus F)\cap\left\{|x|<B^{\frac{p_0'}{n}}\right\}} \left(\frac{1}{2}A_3 B|x|^{-n}\right)^{p_0} dx \\
&\geq \int_{\left\{A_5(|F|+A_2^n)^{\frac{1}{n}}<|x|<B^{\frac{p_0'}{n}}\right\}\cap G} \left(\frac{1}{2}A_3 B|x|^{-n}\right)^{p_0} dx \\
&= \left(\frac{A_3 B}{2}\right)^{p_0} \sigma(\Lambda) \int_{A_5(|F|+A_2^n)^{\frac{1}{n}}}^{B^{\frac{p_0'}{n}}} t^{-np_0+n-1} dt \\
&= \left(\frac{A_3 B}{2}\right)^{p_0} \frac{\sigma(\Lambda)}{n(1-p_0)} \left[B^{p_0'(1-p_0)} - A_5^{n(1-p_0)}(|F| + A_2^n)^{1-p_0}\right].
\end{aligned}$$

So there exists A_6 such that

$$(|F| + A_2^n)^{1-p_0} \leq A_6^{1-p_0} B^{p_0'(1-p_0)}.$$

Therefore, if

$$B > (2A_2^n A_6^{-1})^{\frac{1}{p_0'}},$$

then

$$|F| \geq A_6 B^{p_0'} - A_2^n \geq \frac{A_6 B^{p_0'}}{2}. \tag{2.4.24}$$

Now take $g(x) = \operatorname{sgn}(b(x))\chi_F(x)$. Denote the conjugate operator of T_Ω by T_Ω'. For $x \in Q_1$ we then have that

$$\left|[b, T_\Omega']g(x)\right| \geq \left|\int_F \frac{\Omega(y-x)}{|y-x|^n}|b(y)|dy\right| - |b(x)| \left|\int_{\mathbb{R}^n} \frac{\Omega(y-x)}{|y-x|^n} g(y)dy\right|. \tag{2.4.25}$$

Since $y \in F$, we have

$$|b(y)| > \left(\frac{BA_3}{2A_4}\right)|y|.$$

Moreover, since $x \in Q_1$, applying the same method as in the estimate of (2.4.20), we know that there exists A_7 such that

$$\left|\int_F \frac{\Omega(y-x)}{|y-x|^n}|b(y)|dy\right| \geq A_7 \int_F |y|^{-n} \left(\frac{BA_3}{2A_4}\right) |y| dy.$$

From (2.4.24) it follows that there exists A_8 such that

$$\left|\int_F \frac{\Omega(y-x)}{|y-x|^n}|b(y)|dy\right| \geq A_8 B^{1+\frac{p_0'}{n}}. \tag{2.4.26}$$

On the other hand, there exists A_9 such that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \frac{\Omega(y-x)}{|y-x|^n} g(y)dy\right| &\leq \int_F \frac{|\Omega(y-x)|}{2^n|y|^n}dy \\ &\leq 2^{-n}\int_{A_2<|y|<B^{\frac{p_0'}{n}}} \frac{dy}{|y|^n}\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \\ &\leq A_9 \log B,\end{aligned}$$

which together with (2.4.25) and (2.4.26) yields that

$$\left|[b,T_\Omega']g(x)\right| \geq A_8 B^{1+\frac{p_0'}{n}} - A_9|b(x)|\log B. \tag{2.4.27}$$

Since $[b,T_\Omega']$ is the conjugate operator of $[b,T_\Omega]$, we have

$$\|[b,T_\Omega']\|_{L^{p_0'}\to L^{p_0'}} = 1.$$

So it follows from definition of F and (2.4.27) that

$$\begin{aligned} A_{10}B &\geq \|g\|_{p_0'} \\ &\geq \|[b,T_\Omega']g\|_{p_0'} \\ &\geq \int_{Q_1} \left|[b,T_\Omega']g(x)\right| dx \\ &\geq \int_{Q_1} \left[A_8 B^{1+\frac{p_0'}{n}} - A_9|b(x)|\log B\right] dx \\ &= A_8 B^{1+\frac{p_0'}{n}}|Q_1| - A_9|Q_1|B\log B. \end{aligned} \tag{2.4.28}$$

By (2.4.28), it is clear to have $B \leq A(n,\Omega,p_0)$. Thus we complete the proof of Theorem 2.4.1 (ii).

Theorem 2.4.1 shows that L^p boundedness of the commutator of singular integral operator T_Ω (Ω is a Lipschitz function) can be used to characterize BMO function. Next we will show that L^p boundedness of the commutator of a linear operator T with BMO function can be derived from the weighted L^p boundedness of T.

Theorem 2.4.2 *Suppose that $1 < p < \infty$. The operator T is defined by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy,$$

and the commutator $[b,T]$ of T and b is defined by

$$[b,T]f(x) = \int_{\mathbb{R}^n} K(x,y)[b(x)-b(y)]f(y)dy.$$

For $\omega \in A_s$ $(1 < s < \infty)$, if T is bounded on $L^p(\omega)$, then $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for every $b \in \mathrm{BMO}(\mathbb{R}^n)$.

To prove Theorem 2.4.2, let us first introduce some lemmas.

Lemma 2.4.3 *Let $1 < s < \infty$, $\lambda > 0$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there exists $\eta = \eta(\lambda, s) > 0$, when $\|b\|_{\mathrm{BMO}} < \eta$, such that*

$$e^{\lambda b(x)} \in A_s(\mathbb{R}^n).$$

By the John-Nirenberg inequality, the Reverse Hölder inequality of A_p can be used to prove Lemma 2.4.3. For the details, see Garcia-Cuerva & Rubio de Francia [GaR].

Now we turn back to the proof of Theorem 2.4.2. Take $\lambda = p$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}} < \eta$. By Lemma 2.4.3 we have $e^{pb} \in A_s(\mathbb{R}^n)$. On the other hand, for every $\theta \in [0, 2\pi]$, $b\cos\theta \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|b\cos\theta\|_{\mathrm{BMO}} = \|b\|_{\mathrm{BMO}} < \eta$. Thus

$$e^{pb\cos\theta} \in A_s(\mathbb{R}^n).$$

Now for $z \in \mathbb{C}$,

$$g(z) = e^{z[b(x)-b(y)]}$$

is analytic on $\mathbb{C}$. Thus by the Cauchy integral formula we get

$$\begin{aligned} b(x) - b(y) = g'(0) &= \frac{1}{2\pi i}\int_{|z|=1} \frac{g(z)}{|z|^2}dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}[b(x)-b(y)]}e^{-i\theta}d\theta. \end{aligned} \tag{2.4.29}$$

For $\theta \in [0, 2\pi]$, let

$$h_\theta(x) = f(x)e^{-b(x)e^{i\theta}}.$$

Since $f \in L^p$, we have

$$h_\theta \in L^p(e^{pb\cos\theta}) \quad \text{and} \quad \|h_\theta\|_{L^p(e^{pb\cos\theta})} = \|f\|_{L^p}.$$

Applying (2.4.29), it implies that

$$\begin{aligned}
[b,T](f)(x) &= \int_{\mathbb{R}^n} K(x,y)\left[\frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}[b(x)-b(y)]}e^{-i\theta}d\theta\right] f(y)dy \\
&= \frac{1}{2\pi}\int_0^{2\pi} T(h_\theta)(x)e^{b(x)e^{i\theta}}e^{-i\theta}d\theta.
\end{aligned}$$

Applying Minkowski's inequality and the $L^p(\omega)$ boundedness of T, we have

$$\|[b,T]f\|_{L^p} \le \frac{1}{2\pi}\int_0^{2\pi} \|T(h_\theta)\|_{L^p(e^{pb\cos\theta})}d\theta \le C\|f\|_{L^p}.$$

This finishes the proof of Theorem 2.4.2.

We can immediately obtain the L^p boundedness of commutator $[b, T_\Omega]$ by Theorem 2.4.2 and Theorem 2.3.9.

Corollary 2.4.1 *Suppose that* $\Omega \in L^q(\mathbb{S}^{n-1})$ $(1 < q \le \infty)$ *satisfies (2.2.1) and (2.2.2). If* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *then* $[b, T_\Omega]$ *is bounded on* L^p *for* $1 < p < \infty$.

Remark 2.4.2 *Corollary 2.4.1 improves the conclusion (i) of Theorem 2.4.1.*

Next we will give the weighted boundedness of commutator $[b, T_\Omega]$. In fact, similar to the conclusion of Theorem 2.4.2, weighted L^p boundedness of $[b, T_\Omega]$ can also be attribute to weighted L^p boundedness of T_Ω.

Theorem 2.4.3 *Let* $1 < p < \infty$. *Suppose that* T *is defined as in Theorem 2.4.2 and satisfies* $\|Tf\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$ *for* $\omega \in A_s$ $(1 < s < \infty)$. *Then* $[b,T]$ *is bounded on* $L^p(\omega)$.

Proof. The idea in the proof is essentially the same as that of Theorem 2.4.2.

Since $\omega \in A_s$, by Proposition 1.4.2 (x) we know that there exists $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in A_s$. So

$$\|Tf\|_{L^p(\omega^{1+\varepsilon})} \le C\|f\|_{L^p(\omega^{1+\varepsilon})}. \tag{2.4.30}$$

Let $\lambda = \frac{p(1+\varepsilon)}{\varepsilon}$. By Lemma 2.4.3 we have that there is $\eta > 0$ such that

$$e^{pb(x)(1+\varepsilon)/\varepsilon} \in A_s(\mathbb{R}^n)$$

whenever $\|b\|_{\mathrm{BMO}} < \eta$. Thus for every $\theta \in [0, 2\pi]$,

$$e^{pb(1+\varepsilon)\cos\theta/\varepsilon} \in A_s(\mathbb{R}^n)$$

still holds. By the weighted boundedness of T we get

$$\|Tf\|_{L^p(e^{pb(1+\varepsilon)\cos\theta/\varepsilon})} \le C\|f\|_{L^p(e^{pb(1+\varepsilon)\cos\theta/\varepsilon})}. \tag{2.4.31}$$

Applying the Stein-Weiss interpolation theorem with change of measure (Lemma 2.2.5) to (2.4.30) and (2.4.31) we have

$$\|Tf\|_{L^p(\omega e^{pb\cos\theta})} \le C\|f\|_{L^p(\omega e^{pb\cos\theta})}. \tag{2.4.32}$$

Now for $\theta \in [0, 2\pi]$, denote $h_\theta = fe^{-be^{i\theta}}$. Then by $f \in L^p(\omega)$ we know

$$h_\theta \in L^p(\omega e^{pb\cos\theta}) \quad \text{and} \quad \|h_\theta\|_{L^p(\omega e^{pb\cos\theta})} = \|f\|_{L^p(\omega)}. \tag{2.4.33}$$

It follows from (2.4.29) that

$$[b, T](f)(x) = \frac{1}{2\pi}\int_0^{2\pi} T(h_\theta)(x)e^{b(x)e^{i\theta}}e^{-i\theta}d\theta.$$

Thus by Minkowski's inequality and (2.4.32) as well as (2.4.33), we have

$$\|[b, T]f\|_{L^p(\omega)} \le \frac{1}{2\pi}\int_0^{2\pi} \|T(h_\theta)\|_{L^p(\omega e^{pb\cos\theta})}d\theta \le C\|f\|_{L^p(\omega)}.$$

This finishes the proof of Theorem 2.4.3. ∎

Applying Theorem 2.4.3 and Theorem 2.3.9, it is easy to get the weighted L^p boundedness of commutator $[b, T_\Omega]$.

Theorem 2.4.4 *Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ $(q > 1)$ satisfies (2.2.1) and (2.2.2). If $b \in \mathrm{BMO}(\mathbb{R}^n)$ and p, q, ω satisfy one of the following conditions, then $[b, T_\Omega]$ is bounded on $L^p(\omega)$:*

(i) $q' \le p < \infty$, $p \ne 1$ *and* $\omega \in A_{\frac{p}{q'}}$*;*

(ii) $1 < p \le q$, $p \ne \infty$ *and* $\omega^{1-p'} \in A_{\frac{p'}{q'}}$*;*

(iii) $1 < p < \infty$ *and* $\omega^{q'} \in A_p$*.*

Proof. The result under the condition (i) is a direct consequence of Theorem 2.4.3 and Theorem 2.3.9. Using the conclusion under the condition (i) and dual method (see the proof of Theorem 2.2.3) we can obtain the conclusion under the condition (ii). The conclusion under the condition (iii) is a corollary of the conclusions under the condition (i) and (ii) (for the details, see the proof of Theorem 2.2.4). ■

Remark 2.4.3

1. Using the method in the proof of Theorem 2.4.2 and Theorem 2.4.3 as well as induction, we can get the similar result about commutators of degree k of linear operator T and b. Here the commutator of degree k is defined by

$$[b,T]^k f(x) = [b,\cdots,[b,T]\cdots]f(x) = \int_{\mathbb{R}^n} K(x,y)[b(x)-b(y)]^k f(y)dy.$$

2. In the same way, we can obtain the weighted L^p boundedness of commutator of degree k generated by a singular integral operator with rough kernel T_Ω and a function b.

2.5 Notes and references

Lemma 2.1.1 is due to Cotlar [Cot]. Lemma 2.1.4 and Theorem 2.1.9 are due to Benedek, Calderón and Panzone [BeCP]. Theorem 2.1.11 comes from Garcia-Cuerva and Rubio de Francia [GaR].

The idea of the proof in Lemma 2.2.1 comes from Duoandikoetxea and Rubio de Francia [DuR]. The idea on the decomposition in the proof of Theorem 2.2.2 comes from Christ and Rubio de Francia [ChrR]. For L^q-Dini condition in Theorem 2.2.3 and Lemma 2.2.2, we refer to Kurtz and Wheeden [KuW].

Lemma 2.3.1 is due to Colzani [Col]. Lemma 2.3.2 was first obtained by Stefanov [Ste]. Here the idea of the proof ia taken from Pan, Wu and Yang [PaWY]. For the proof of Lemma 2.3.3, see Grafakos and Stefanov [GrS]. Theorem 2.3.5 was first obtained by Connett [Con] and independently by Ricci and Weiss [RiW]. The proof of Theorem 2.3.5 given here comes from Grafakos and Stefanov [GrS]. Theorem 2.3.6 is due to Grafakos and Stefanov [GrS]. Generalizations of Theorem 2.3.5 and Theorem 2.3.6 to rough kernels supported by subvarieties were established by Fan and Pan [FaP2]. The idea of proving Theorem 2.3.8 is taken from Duoandikoetxea [Du1]. And Theorem 2.3.9 is obtained by Watson [Wat] and independently by Duoandikoetxea [Du1]. In 2003, Ding and Lin [DiLin] established weighted L^p boundedness for the rough operators T_Ω, M_Ω and T^*_Ω with different weights.

In 1976, Coifman, Rochberg and Weiss [CoiRW] established the conclusion (i) in Theorem 2.4.1 for singular integral operators with standard Calderón-Zygmund kernels and the conclusion (ii) in Theorem 2.4.1 for the Riesz transforms $R_j, 1 \leq j \leq n$. Theorem 2.4.1 is obtained by Janson [Ja] and independently by Uchiyama [Uc]. The proof of Theorem 2.4.1 given here comes from [Uc]. Theorem 2.4.2 is first established by Alvarez, Bagby, Kurtz and Perez [ABKP]. The proof of Theorem 2.4.2 here comes from Hu [Hu] and Ding, Lu [DiL4]. In addition, Corollary 2.4.1 has been improved by Lu and Wu [LuW3]in which the condition $\Omega \in L^q(\mathbb{S}^{n-1})(q > 1)$ in Corollary 2.4.1 was replaced by a class of bigger block spaces $B_q^{0,0}(\mathbb{S}^{n-1}), 1 < q < \infty$. In fact, $\bigcup_{r>1} L^r(\mathbb{S}^{n-1}) \subset B_q^{0,0}(\mathbb{S}^{n-1}), 1 < q < \infty$ (see Lu, Taibleson and Weiss [LuTW]).

Chapter 3

Fractional Integral Operators

In this chapter, we will investigate the boundedness of the Riesz potential and its general form (fractinal integral). If $0 < \alpha < n$, then $|x|^{-n+\alpha} \in L_{\mathrm{loc}}(\mathbb{R}^n)$. Thus under the sense of distribution, we have that

$$\widehat{\left(\frac{1}{|x|^{n-\alpha}}\right)}(\xi) = \gamma(\alpha)(2\pi)^{-\alpha}|\xi|^{-\alpha}, \tag{3.0.1}$$

where $\gamma(\alpha) = \pi^{\frac{n}{2}} 2^{\alpha}\Gamma\left(\frac{\alpha}{2}\right)/\Gamma\left(\frac{n-2}{2}\right)$.

The Riesz potential is an operator defined by

$$I_\alpha(f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = \frac{1}{\gamma(\alpha)} \left(\frac{1}{|\cdot|^{n-\alpha}} * f\right)(x). \tag{3.0.2}$$

Let us now explain the relationship between the Riesz potential and the Laplacian operator of fractional degree. Let $\triangle$ be the Laplacian operator, that is,

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

For any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, by the property of Fourier transform, we have

$$\widehat{(-\triangle\varphi)}(\xi) = 4\pi^2|\xi|^2\widehat{\varphi}(\xi) = (2\pi|\xi|)^2\widehat{\varphi}(\xi).$$

Thus, for $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $0 < \alpha < n$, it follows from (3.0.1) and (3.0.2) that

$$\widehat{(I_\alpha \varphi)}(\xi) = (2\pi|\xi|)^{-\alpha}\widehat{\varphi}(\xi) = ((-\triangle)^{-\alpha/2}\widehat{\varphi)}(\xi).$$

Note that under the sense of distribution, for $f \in \mathscr{S}(\mathbb{R}^n)$,

$$f(x) = \mathrm{I}_1 * \left(\sum_{j=1}^{n} R_j \partial_j f\right)(x) = \sum_{j=1}^{n} (\mathrm{I}_1 * R_j \partial_j f)(x), \tag{3.0.3}$$

where R_j denote the Riesz transforms (see the definition in chapter 2).

By the L^p-boundedness of the Riesz transform R_j $(1 < p < \infty, j = 1, 2, \cdots, n)$ and the (L^p, L^q)-boundedness of the Riesz potential with 1-order, from $\partial_j f \in L^p(1 < p < n, j = 1, 2, \cdots, n)$ and (3.0.3), it follows that $f \in L^q(\mathbb{R}^n)$. Therefore the research to the Riesz potential is closely related to the theory of Sobolev space and the Laplacian operator of fracrional degree.

3.1 Riesz potential

In this chapter we will show the (L^p, L^q)-boundedness of the Riesz potential $I_\alpha, 0 < \alpha < n$. We first show that

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \tag{3.1.1}$$

is a necessary condition for ensuring

$$\|I_\alpha f\|_q \le C\|f\|_p. \tag{3.1.2}$$

Indeed, for $\delta > 0$, we define the dilation transform

$$\tau_\delta(f)(x) = f(\delta x).$$

Then, for $0 < \alpha < n$,

$$\tau_{\delta^{-1}} I_\alpha \tau_\delta = \delta^{-\alpha} I_\alpha \tag{3.1.3}$$

and

$$\|\tau_\delta(f)\|_p = \delta^{-n/p}\|f\|_p, \ \|\tau_{\delta^{-1}} I_\alpha(f)\|_q = \delta^{n/q}\|I_\alpha(f)\|_q. \tag{3.1.4}$$

Thus, for $\delta > 0$, it follows from (3.1.2), (3.1.3), and (3.1.4) that

$$\begin{aligned} \|I_\alpha f\|_q &= \delta^\alpha \|\tau_{\delta^{-1}} I_\alpha \tau_\delta(f)\|_q \\ &= \delta^{\alpha+n/q} \|I_\alpha \tau_\delta(f)\|_q \\ &\le C\delta^{\alpha+n/q} \|\tau_\delta f\|_p \\ &= C\delta^{\alpha+n/q-n/p} \|f\|_p. \end{aligned} \tag{3.1.5}$$

Since (3.1.5) holds for all $\delta > 0$, we conclude that (3.1.1) must hold.

The following theorem shows that, when $1 < p < \frac{n}{\alpha}$, (3.1.1) is also a sufficient condition for ensuring (3.1.2). We first formulate a lemma.

Lemma 3.1.1 *Let $f \in L^p(\mathbb{R}^n), 1 \le p < \frac{n}{\alpha}$. Then, for $x \in \mathbb{R}^n$, we have*

$$|I_\alpha f(x)| \le C\|f\|_p^{\frac{\alpha p}{n}} \Big[Mf(x)\Big]^{1-\frac{\alpha p}{n}},$$

where M is the Hardy-Littlewood maximal function and $C = C(\alpha, p, n)$.

Proof. Fix $x \in \mathbb{R}^n$, for some $r > 0$,

$$|I_\alpha f(x)| \le \int_{|y|\le r} \frac{f(x-y)}{|y|^{n-\alpha}} dy + \int_{|y|>r} \frac{|f(x-y)|}{|y|^{n-\alpha}} dy := J_1 + J_2.$$

For J_1, we write

$$\begin{aligned} J_1 &= \sum_{j=0}^{\infty} \int_{2^{-j-1}r<|y|\le 2^{-j}r} \frac{f(x-y)}{|y|^{n-\alpha}} dy \\ &\le \sum_{j=0}^{\infty} \frac{1}{(2^{-j-1}r)^{n-\alpha}} \int_{2^{-j-1}r<|y|\le 2^{-j}r} |f(x-y)| dy \\ &\le Cr^\alpha \sum_{j=0}^{\infty} \frac{(2^{-j})^\alpha}{(2^{-j}r)^n} \int_{|y|\le 2^{-j}r} |f(x-y)| dy \\ &\le Cr^\alpha Mf(x). \end{aligned} \tag{3.1.6}$$

As to J_2, if $p = 1$, then

$$J_2 \le r^{\alpha-n} \|f\|_1. \tag{3.1.7}$$

If $1 < p < \frac{n}{\alpha}$, then Hölder's inequality implies that

$$\begin{aligned} J_2 &\le \left(\int_{|y|>r} |y|^{(\alpha-n)p'} dy \right)^{1/p'} \|f\|_p \\ &\le Cr^{\alpha-\frac{n}{p}} \|f\|_p. \end{aligned} \tag{3.1.8}$$

Therefore, when $1 \le p < \frac{n}{\alpha}$, from (3.1.6)- (3.1.8), for all $x \in \mathbb{R}^n$, it follows that

$$|I_\alpha f(x)| \le C(r^\alpha Mf(x) + r^{\alpha-\frac{n}{p}}\|f\|_p).$$

Now take $r = (\frac{\|f\|_p}{Mf(x)})^{\frac{p}{n}}$, then

$$r^\alpha Mf(x) = r^{\alpha-\frac{n}{p}}\|f\|_p = \|f\|_p^{\frac{\alpha p}{n}} Mf(x)^{1-\frac{\alpha p}{n}}.$$

Thus the lemma is proved. ∎

Theorem 3.1.1 (Hardy-Littlewood-Sobolev theorem) *Suppose that* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

(i) If $f \in L^p(\mathbb{R}^n)(1 < p < \frac{n}{\alpha})$, *then* $\|I_\alpha f\|_q \le C\|f\|_p$ *;*

(ii) If $f \in L^1(\mathbb{R}^n)$, *then for all* $\lambda > 0$, $|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| \le (\frac{C}{\lambda}\|f\|_1)^{\frac{n}{n-\alpha}}$, *where* $C = C(\alpha, n, p)$.

Proof. (i) Note that $(1 - \frac{\alpha p}{n})q = p$. By Lemma 3.1.1 and L^p-boundedness of the Hardy-Littlewood maximal operator M for $1 < p < \infty$, we get

$$\|I_\alpha f\|_q \le C\|f\|_p^{\frac{\alpha p}{n}}\|Mf\|_p^{1-\frac{\alpha p}{n}} \le C\|f\|_p.$$

(ii) Using Lemma 3.1.1 again and the weak $(1,1)$-boundedness of the operator M, we have that

$$\begin{aligned}
|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| &= \left|\left\{x \in \mathbb{R}^n : Mf(x) > \left(\frac{\lambda}{C\|f\|_1^{\frac{\alpha}{n}}}\right)^{\frac{n}{n-\alpha}}\right\}\right| \\
&\le C_1\left(\frac{C\|f\|_1^{\frac{\alpha}{n}}}{\lambda}\right)^{\frac{n}{n-\alpha}}\|f\|_1 \\
&\le \left(\frac{C}{\lambda}\|f\|_1\right)^{\frac{n}{n-\alpha}}.
\end{aligned}$$

∎

Let us give some remarks on the Riesz potential I_α.

Remark 3.1.1 *The second result (ii) of Theorem 3.1.1 cannot be improved as*

$$\|I_\alpha f\|_{\frac{n}{n-\alpha}} \le C\|f\|_1. \tag{3.1.9}$$

There is a counterexample for (3.1.9). Let $f(y) = 1$ *when* $|y| \le 1$ *and* $f(y) = 0$ *when* $|y| > 1$. *Then*

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{|y|\le 1} \frac{dy}{|x-y|^{n-\alpha}}.$$

For $|x| > 1$ *and* $|y| \le 1$, $|x-y| \le |x| + |y| \le |x| + 1 < 2|x|$, *and hence*

$$|I_\alpha f(x)| \ge \frac{C}{|x|^{n-\alpha}} \qquad \textit{for} \quad |x| > 1,$$

which implies

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^{\frac{n}{n-\alpha}} dx \ge C \int_{|x|>1} \frac{dx}{|x|^n} = \infty.$$

Remark 3.1.2 *When* $p = \frac{n}{\alpha}$, *the first result (i) of Theorem 3.1.1 is not true. Example:*

$$f(x) = \begin{cases} |x|^{-\alpha} \left(\log \frac{1}{|x|}\right)^{-\frac{\alpha}{n}(1+\varepsilon)}, & |x| \le \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$$

where ε *is sufficiently small. Obviously,* $f \in L^{\frac{n}{\alpha}}(\mathbb{R}^n)$. *But if we take any* ε *satisfies* $\frac{\alpha}{n}(1+\varepsilon) \le 1$, *then*

$$I_\alpha(f)(0) = \frac{1}{\gamma(\alpha)} \int_{|x|\le\frac{1}{2}} |x|^{-n} \left(\log \frac{1}{|x|}\right)^{-\frac{\alpha}{n}(1+\varepsilon)} dx = \infty,$$

and $I_\alpha(f)$ *is not essentially bounded near the origin. Hence* $I_\alpha f(x) \notin L^\infty(\mathbb{R}^n)$.

3.2 Weighted boundedness of Riesz potential

To study the weighted boundedness of I_α, we need to introduce the following fractional maximal operator M_α. For $0 < \alpha < n$ and $f \in L_{loc}(\mathbb{R}^n)$, define M_α by

$$M_\alpha(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|\le r} |f(x-y)| dy. \tag{3.2.1}$$

An equivalent definition of M_α is

$$M_\alpha(f)(x) = \sup_{Q_x} \frac{1}{|Q_x|^{1-\frac{\alpha}{n}}} \int_{Q_x} |f(y)|dy, \tag{3.2.2}$$

where the supremum is taken over all cubes Q_x in $\mathbb{R}^n$ with the center at x and with the sides parallel to the axes.

The fractional maximal operator M_α will be dominated by I_α in some sense. That is, for $0 < \alpha < n$, $f \in L_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$M_\alpha(f)(x) \le \gamma(\alpha) I_\alpha(|f|)(x). \tag{3.2.3}$$

In fact, for fixed $x \in \mathbb{R}^n$ and $r > 0$, we have that

$$\begin{aligned} I_\alpha(|f|)(x) &= \tfrac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{|f(x-y)|}{|y|^{n-\alpha}} dy \\ &\ge \tfrac{1}{\gamma(\alpha)} \int_{|y|\le r} \frac{|f(x-y)|}{|y|^{n-\alpha}} dy \\ &\ge \tfrac{1}{\gamma(\alpha)} \tfrac{1}{r^{n-\alpha}} \int_{|y|\le r} |f(x-y)|dy. \end{aligned} \tag{3.2.4}$$

The desired consequence follows from taking supremum for $r > 0$ on both sides of (3.2.4).

Theorem 3.2.1 *Suppose that* $0 < \alpha < n$, $1 \le p \le \frac{n}{\alpha}$ *and* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$.

(i) If $f \in L^p(\mathbb{R}^n)$ $\left(1 < p \le \dfrac{n}{\alpha}\right)$, *then*

$$\|M_\alpha f\|_q \le C\|f\|_p.$$

(ii) If $f \in L^1(\mathbb{R}^n)$, *then for any* $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}| \le \left(\frac{C}{\lambda}\|f\|_1\right)^{\frac{n}{n-\alpha}}.$$

The above constant C *only depends on* α, n, p.

Proof. From (3.2.3) and Theorem 3.1.1, when $1 \le p < \frac{n}{\alpha}$, Theorem 3.2.1 follows immediately. For $p = \frac{n}{\alpha}$, Hölder's inequality implies that M_α is bounded from $L^{\frac{n}{\alpha}}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. ■

Let us now consider the weighted boundedness of the fractional maximal operator M_α. First we formulate a definition on the class of $A(p, q)$ and the relation between the class of $A(p,q)$ and the class of A_p.

Suppose that $\omega(x)$ is a nonnegative locally integrable function on $\mathbb{R}^n$. Define $\omega \in A(p,q)(1<p,q<\infty)$, if there exists a constant $C>0$, such that for any cube Q in $\mathbb{R}^n$,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)^q dx\right)^{\frac{1}{q}} \left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'} dx\right)^{\frac{1}{p'}} \le C < \infty; \tag{3.2.5}$$

and say $\omega \in A(1,q)(1<q<\infty)$, if there exists a constant $C>0$, such that for any cube Q,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)^q dx\right)^{\frac{1}{q}} \left(\operatorname*{ess\,sup}_Q \frac{1}{\omega(x)}\right) \le C < \infty. \tag{3.2.6}$$

Theorem 3.2.2 (Relation between $A(p,q)$ and A_p) *Suppose that $0<\alpha<n$, $1\le p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.*

(i) If $p>1$, then $\omega\in A(p,q) \iff \omega^q \in A_{q\frac{n-\alpha}{n}} \iff \omega^q\in A_{1+\frac{q}{p'}} \iff \omega^{-p'}\in A_{1+\frac{p'}{q}}$;

(ii) If $p>1$, then $\omega\in A(p,q) \Longrightarrow \omega^q\in A_q$ and $\omega^p\in A_p$;

(iii) If $p=1$, then $\omega\in A(1,q) \iff \omega^q\in A_1$.

Theorem 3.2.3 *Suppose that $0<\alpha<n$, $1\le p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{n}{\alpha}$ and $\omega\in A(p,q)$. Then for $\forall\lambda>0$, there exists a constant $C>0$ such that for $\forall f\in L^p(\mathbb{R}^n,\omega^p)$,*

$$\left(\int_{\{x\in\mathbb{R}^n: M_\alpha f(x)>\lambda\}} \omega(x)^q dx\right)^{\frac{1}{q}} \le \frac{C}{\lambda}\left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}. \tag{3.2.7}$$

Proof. For $\lambda>0$ and $K>0$, set

$$E_\lambda = \{x\in\mathbb{R}^n : M_\alpha f(x)>\lambda\},$$
$$E_{\lambda,K} = E_\lambda \cap B(0,K), \text{ where } B(0,K)=\{x\in\mathbb{R}^n : |x|<K\}.$$

Thus, for $\forall x\in E_{\lambda,K}$, by the definition of M_α, there exists a Q_x such that

$$|Q_x|^{-1+\frac{\alpha}{n}}\int_{Q_x}|f(y)|dy>\lambda. \tag{3.2.8}$$

Since $E_{\lambda,K} \subset \bigcup_{x \in E_{\lambda,K}} Q_x$, there exists $\{x_j\} \subset E_{\lambda,K}$, such that $E_{\lambda,K} \subset \bigcup_j Q_{x_j}$. $\{Q_{x_j}\}$ is bounded overlapping, i.e., $\exists\ C = C(n)$, we have $\sum_j \chi_{Q_{x_j}}(x) \leq C(n), \forall x \in \mathbb{R}^n$. (Here we apply the Besicovitch overlapping theorem, to see [GaR]). Thus,

$$\left(\int_{E_{\lambda,K}} \omega(y)^q dy\right)^{\frac{p}{q}} \leq \left(\sum_j \int_{Q_{x_j}} \omega(y)^q dy\right)^{\frac{p}{q}}. \tag{3.2.9}$$

For $\frac{p}{q} < 1$, the right side of (3.2.9) is dominated by

$$\sum_j \left(\int_{Q_{x_j}} \omega(y)^q dy\right)^{\frac{p}{q}}. \tag{3.2.10}$$

Since all Q_{x_j} satisfy (3.2.8), combining (3.2.9) with (3.2.10) yields that

$$\left(\int_{E_{\lambda,K}} \omega(y)^q dy\right)^{\frac{p}{q}} \leq \sum_j \left(\int_{Q_{x_j}} \omega(y)^q dy\right)^{\frac{p}{q}} \left(\frac{1}{\lambda |Q_{x_j}|^{1-\frac{\alpha}{n}}} \int_{Q_{x_j}} |f(y)| dy\right)^p. \tag{3.2.11}$$

If $p > 1$, applying Hölder's inequality and (3.2.5), we conclude that

$$\begin{aligned}\left(\int_{E_{\lambda,K}} \omega(y)^q dy\right)^{\frac{p}{q}} &\leq \sum_j \left(\int_{Q_{x_j}} \omega(y)^q dy\right)^{\frac{p}{q}} \cdot \lambda^{-p} |Q_{x_j}|^{1-p-\frac{p}{q}} \\ &\quad \times \int_{Q_{x_j}} |f(y)\omega(y)|^p dy \left(\int_{Q_{x_j}} \omega(y)^{-p'} dy\right)^{\frac{p}{p'}} \\ &\leq C\lambda^{-p} \sum_j \int_{Q_{x_j}} |f(y)\omega(y)|^p dy \\ &\leq C\lambda^{-p} \int_{\mathbb{R}^n} |f(x)\omega(y)|^p dy.\end{aligned} \tag{3.2.12}$$

The last inequality of (3.2.12) is due to the property of $\{Q_{x_j}\}$ having finite overlap. Note that the constant C in (3.2.12) is independent of K, so the monotone convergence theorem leads to (3.2.7).

If $p = 1$, then

$$\begin{aligned}\int_{Q_{x_j}} |f(y)|dy &= \int_{Q_{x_j}} |f(y)\omega(y)|\omega(y)^{-1}dy \\ &\leq \operatorname*{ess\,sup}_{Q_{x_j}} \omega(y)^{-1} \int_{Q_{x_j}} |f(y)\omega(y)|dy.\end{aligned} \tag{3.2.13}$$

By (3.2.11), (3.2.13) and (3.2.6), we have that

$$\begin{aligned}\left(\int_{E_{\lambda,K}} \omega(y)^q dy\right)^{\frac{1}{q}} &\leq \sum_j \frac{1}{\lambda}\left(\int_{Q_{x_j}} \omega(y)^q dy\right)^{\frac{1}{q}} \frac{1}{|Q_{x_j}|^{\frac{1}{q}}} \\ &\quad \times \operatorname*{ess\,sup}_{Q_{x_j}} \frac{1}{\omega(y)} \left(\int_{Q_{x_j}} |f(y)\omega(y)|dy\right) \\ &\leq C\frac{1}{\lambda}\sum_j \int_{Q_{x_j}} |f(y)\omega(y)|dy \\ &\leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(y)w(y)|dy.\end{aligned}$$

Thus, let $k \to \infty$, we obtain that (3.2.7) still holds for $p = 1$. ■

Theorem 3.2.3 shows that $A(p, q)$ is a sufficient condition for ensuring that M_α is weighted weak (p, q)-bounded. The following theorem will illustrate that $A(p, q)$ is also sufficient for M_α being a weighted (p, q)-bounded operator with $1 < p < \frac{n}{\alpha}$.

Theorem 3.2.4 *Suppose that* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $\omega \in A(p, q)$, *then*

$$\left(\int_{\mathbb{R}^n} [M_\alpha f(x)\omega(x)]^q \, dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}, \tag{3.2.14}$$

where C *is independent of* f.

Proof. Since $\omega \in A(p, q)$, Theorem 3.2.2 (i) leads to $\omega^q \in A_{1+\frac{q}{p'}}$. By the elementary property of A_p-weight, there exists $1 < s < 1 + \frac{q}{p'}$ such that $\omega^q \in A_s$. Now we take p_1, q such that

$$1 < p_1 < p, \ \frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n} \ \text{ and } \ s = 1 + \frac{q_1}{p_1'}. \tag{3.2.15}$$

From (3.2.15) and (i) in Theorem 3.2.2, it follows that $\omega^{\frac{q}{q_1}} \in A(p_1, q_1)$. Using Theorem 3.2.3, for any $\lambda > 0$, we have that

$$\left(\int_{E_\lambda} \left[\omega(x)^{\frac{q}{q_1}}\right]^{q_1} dx\right)^{\frac{p_1}{q_1}} \leq C\lambda^{-p_1} \left(\int_{\mathbb{R}^n} |f(x)|^{p_1} \left(\omega(x)^{\frac{q}{q_1}}\right)^{p_1} dx\right).$$

That is equivalent to

$$\left(\int_{E_\lambda} v(x)dx\right)^{\frac{p_1}{q_1}} \leq C\lambda^{-p_1} \int_{\mathbb{R}^n} |f(x)|^{p_1} v(x)^{\frac{p_1}{q_1}} dx, \tag{3.2.16}$$

where $v(x) = \omega^q(x)$.

Now let $f(x) = g(x)v(x)^{\frac{\alpha}{n}}$ and define a sublinear operator

$$T(g)(x) = M_\alpha(gv^{\frac{\alpha}{n}})(x).$$

Thus (3.2.16) is equivalent to

$$\int_{\{x\in\mathbb{R}^n : T(g)(x)>\lambda\}} v(x)dx \leq C\lambda^{-q_1} \left(\int_{\mathbb{R}^n} |g(x)|^{p_1} v(x)dx\right)^{\frac{q_1}{p_1}}. \tag{3.2.17}$$

On the other hand, we take p_2 such that $p < p_2 < \frac{n}{\alpha}$ and let $\frac{1}{q_2} = \frac{1}{p_2} - \frac{\alpha}{n}$, then

$$1 + \frac{q_2}{p_2'} > 1 + \frac{q}{p'}.$$

By Theorem 3.2.2 (i), it implies that $\omega^{\frac{q}{q_2}} \in A(p_2, q_2)$. Similar to (3.2.17), we have

$$\int_{\{x\in\mathbb{R}^n : T(g)(x)>\lambda\}} v(x)dx \leq C\lambda^{-q_2} \left(\int_{\mathbb{R}^n} |g(x)|^{p_2} v(x)dx\right)^{\frac{q_2}{p_2}}. \tag{3.2.18}$$

Applying the Marcinkiewicz interpolation theorem to (3.2.17) and (3.2.18), we obtain that

$$\left(\int_{\mathbb{R}^n} [T(g)(x)]^q v(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |g(x)|^p v(x)dx\right)^{\frac{1}{p}}. \tag{3.2.19}$$

Set $g(x) = f(x)v(x)^{\frac{-\alpha}{n}}$ and $v(x) = \omega(x)^q$, then we have that

$$\left(\int_{\mathbb{R}^n} [M_\alpha(f)(x)\omega(x)]^q dx\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.$$

This finishes the proof of Theorem 3.2.4. ■

From the weighted boundedness of M_α, we can deduce the weighted boundedness of the Riesz potential I_α. Before stating the next lemma, we will give a definition on A_∞. A nonnegative function $\omega(x)$ on $\mathbb{R}^n$ satisfies A_∞ condition, if for a given $\varepsilon > 0$, there exists a $\delta > 0$, such that for any cube Q, and a measurable subset E of Q satisfying $|E| \le \delta|Q|$,

$$\int_E \omega(x)dx \le \varepsilon \int_Q \omega(x)dx. \tag{3.2.20}$$

We call all functions satisfying the A_∞ condition as the class of A_∞. The relation between A_∞ and A_p $(1 \le p < \infty)$ is

$$A_\infty = \bigcup_{p\ge 1} A_p. \tag{3.2.21}$$

The next lemma will reveal another relation between I_α and M_α. For its proof, we refer to [MuW].

Lemma 3.2.1 *Suppose that* $0 < \alpha < n$, $0 < q < \infty$ *and* $\omega(x) \in A_\infty$. *Then there exists a constant* C *independent of* f *such that*

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^q \omega(x)dx \le C \int_{\mathbb{R}^n} [M_\alpha f(x)]^q \omega(x)dx$$

and

$$\sup_{\lambda>0} \lambda^q \int_{\{x:|I_\alpha f(x)|>\lambda\}} \omega(x)dx \le C \sup_{\lambda>0} \lambda^q \int_{\{x:M_\alpha f(x)>\lambda\}} \omega(x)dx.$$

Applying Lemma 3.2.1, we can get the weighted boundedness of I_α.

Theorem 3.2.5 *Suppose that* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $\omega(x) \in A(p,q)$.

(i) If $1 < p < \infty$, *then*

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)\omega(x)|^q dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}; \tag{3.2.22}$$

(ii) If $p = 1$, *then for any* $\lambda > 0$,

$$\int_{\{x:|I_\alpha f(x)|>\lambda\}} \omega(x)^q dx \le \frac{C}{\lambda^q} \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|dx\right)^q, \tag{3.2.23}$$

where C *is independent of* f, λ.

The proof of Theorem 3.2.5 is simple. In fact, by Theorem 3.2.2, $\omega(x)^q \in A_{1+\frac{q}{p'}}$ $(p>1)$ or $\omega(x)^q \in A_1$ $(p=1)$, therefore by (3.2.21), we have $\omega(x)^q \in A_\infty$. Applying Lemma 3.2.1, Theorem 3.2.3 and Theorem 3.2.4, we obtain that both (3.2.22) and (3.2.23) hold.

3.3 Fractional integral operator with homogeneous kernels

In this chapter we will discuss the (L^p, L^q)-boundedness and weighted boundedness of fractional integral operators which is more general than the Riesz potential I_α.

Assume that $\Omega(x)$ is a homogeneous function with degree zero on $\mathbb{R}^n$, i.e. for $\forall \lambda > 0, \forall x \in \mathbb{R}^n$,

$$\Omega(\lambda x) = \Omega(x), \tag{3.3.1}$$

as well as $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$, where $\mathbb{S}^{n-1}$ denotes the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$, $0 < \alpha < n$. Then the fractional integral operator with homogeneous kernel is defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy. \tag{3.3.2}$$

It is obvious that when $\Omega \equiv 1$, $T_{\Omega,\alpha}$ is the same as the Riesz potential I_α except for a constant. On the other hand, if $\alpha = 0$ and Ω satisfies the vanishing moment condition on $\mathbb{S}^{n-1}$:

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{3.3.3}$$

then $T_{\Omega,\alpha}$ becomes a Calderón-Zygmund singular integral operator (in the sense of principal value Cauchy integral).

The following result shows that the Hardy-Littlewood-Sobolev theorem still holds for $T_{\Omega,\alpha}$.

Theorem 3.3.1 *Suppose that* $0 < \alpha < n$, $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ *satisfies (3.3.1).*

(i) If $f \in L^1(\mathbb{R}^n)$, then for $\forall \lambda > 0$,

$$|\{x \in \mathbb{R}^n : |T_{\Omega,\alpha} f(x)| > \lambda\}| \le \left(\frac{C}{\lambda}\|f\|_1\right)^{\frac{n}{n-\alpha}}.$$

(ii) If $f \in L^p(\mathbb{R}^n)(1 < p < \frac{n}{\alpha})$, then $\|T_{\Omega,\alpha} f\|_q \le C\|f\|_p$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $C = C(n, \alpha, p)$.

Proof. We will complete the proof by three steps. Set

$$K(x) = \Omega(x)/|x|^{n-\alpha}$$

and

$$E(s) = \{x \in \mathbb{R}^n : |K(x)| > s\}.$$

First we will show that for $\forall s > 0$,

$$|E(s)| \le A s^{-\frac{n}{n-\alpha}}, \tag{3.3.4}$$

where A depends only on α, n. In fact, by (3.3.1), we have that

$$\begin{aligned}
|E(s)| &\le \frac{1}{s}\int_{E(s)} \frac{|\Omega(x)|}{|x|^{n-\alpha}}dx \\
&= \frac{1}{s}\int_{\mathbb{S}^{n-1}} |\Omega(x')| \int_0^{\left(\frac{|\Omega(x')|}{s}\right)^{\frac{1}{n-\alpha}}} r^{\alpha-1} dr d\sigma(x') \\
&= A s^{-\frac{n}{n-\alpha}},
\end{aligned}$$

where $A = \frac{1}{\alpha}\|\Omega\|^{\frac{n}{n-\alpha}}_{L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})}$.

Next we prove, for $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, that $T_{\Omega,\alpha}$ is of weak type (p, q). Take a fixed $\mu > 0$, let

$$K_1(x) = \operatorname{sgn}(K(x))\,(|K(x)| - \mu)\,\chi_{E(\mu)}(x)$$

and

$$K_2(x) = K(x) - K_1(x).$$

Thus if $p = 1$, then $\|K_2\|_\infty \le \mu$; if $1 < p < \frac{n}{\alpha}$, from (3.3.4) it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} |K_2(x)|^{p'} dx &= p' \int_0^\mu s^{p'-1} |E(s)| ds \\
&\le p' A \int_0^\mu s^{p'-1-\frac{n}{n-\alpha}} ds \\
&= \frac{p' A}{p' - \frac{n}{n-\alpha}} \cdot \mu^{p'-\frac{n}{n-\alpha}} \\
&= \frac{n-\alpha}{n} A q \mu^{\frac{n}{n-\alpha}\frac{p'}{q}}.
\end{aligned}$$

Thus, when $1 \le p < \frac{n}{\alpha}$, we obtain that

$$\|K_2\|_{p'} \le \left(\frac{n-\alpha}{n}Aq\right)^{\frac{1}{p'}} \mu^{\frac{n}{(n-\alpha)q}}. \tag{3.3.5}$$

So Hölder's inequality implies that

$$\|K_2 * f\|_\infty \le \left(\frac{n-\alpha}{n}Aq\right)^{\frac{1}{p'}} \mu^{\frac{n}{(n-\alpha)q}} \|f\|_p.$$

Now for $\forall \lambda > 0$, set μ such that

$$\left(\frac{n-\alpha}{n}Aq\right)^{\frac{1}{p'}} \mu^{\frac{n}{(n-\alpha)q}} \|f\|_p = \frac{\lambda}{2},$$

then

$$\left|\left\{x \in \mathbb{R}^n : |K_2 * f(x)| > \frac{\lambda}{2}\right\}\right| = 0.$$

Thus

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^n : |T_{\Omega,\alpha} f(x)| > \lambda\}\right| \le \left|\{x \in \mathbb{R}^n : |K_1 * f(x)| > \frac{\lambda}{2}\right\}\right| \\ &\le \left(\frac{2}{\lambda}\|K_1 * f\|_p\right)^p. \end{aligned} \tag{3.3.6}$$

It follows from (3.3.4) that

$$\begin{aligned} \int_{\mathbb{R}^n} |K_1(x)|dx &= \int_{E(\mu)} (|K(x)| - \mu)dx \\ &\le \int_0^\infty |E(t+\mu)|dt \\ &\le A \int_\mu^\infty t^{-\frac{n}{n-\alpha}} dt \\ &= \tfrac{\alpha A}{n-\alpha} \mu^{-\frac{\alpha}{n-\alpha}}. \end{aligned} \tag{3.3.7}$$

For $\forall f \in L^\infty(\mathbb{R}^n), \forall x \in \mathbb{R}^n$, by (3.3.7), we conclude that

$$|K_1 * f(x)| \le \|f\|_\infty \int_{\mathbb{R}^n} |K_1(x)|dx \le \frac{\alpha A}{n-\alpha} \mu^{-\frac{\alpha}{n-\alpha}} \|f\|_\infty. \tag{3.3.8}$$

For $\forall f \in L^1(\mathbb{R}^n)$, we have that

$$\|K_1 * f\|_1 \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K_1(x-y)||f(y)|dydx \le \frac{\alpha A}{n-\alpha} \mu^{-\frac{\alpha}{n-\alpha}} \|f\|_1. \tag{3.3.9}$$

Thus (3.3.8) and (3.3.9) show that $T_1 : f \longmapsto K_1 * f$ is (∞,∞)-type and $(1,1)$-type. The Riesz-Thörin theorem leads to that T_1 is also (p,p)-type $(1<p<\infty)$, and

$$\|T_1\|_{(p,p)} \le \frac{\alpha A}{n-\alpha}\mu^{-\frac{\alpha}{n-\alpha}}. \tag{3.3.10}$$

Combining (3.3.6) with (3.3.10) yields that

$$\begin{aligned} |\{x\in\mathbb{R}^n : |T_{\Omega,\alpha}f(x)|>\lambda\}| &\le \left(\frac{2}{\lambda}\frac{\alpha A}{n-\alpha}\mu^{\frac{-\alpha}{n-\alpha}}\|f\|_p\right)^p \\ &= C\left(\frac{1}{\lambda}\|f\|_p\right)^q, \end{aligned} \tag{3.3.11}$$

where C is independent of λ and f.

Finally we will finish the proof of Theorem 3.3.1. (3.3.11) tells us that consequence (i) holds. Now we prove that (ii) also holds. For $\forall\, 1<p<\frac{n}{\alpha}$, set p_0 such that $p<p_0<\frac{n}{\alpha}$, and let q_0 satisfy

$$\frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{n}.$$

Thus by the second step we know that $T_{\Omega,\alpha}$ is of weak type (p_0,q_0). By (i) and the Marcinkiewicz interpolation theorem, we see that $T_{\Omega,\alpha}$ is (p,q)-type, where $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. In fact, there exists $0<\theta<1$ so that $\frac{1}{p}=\frac{1-\theta}{p_0}+\theta$ and then $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{(n-\alpha)\theta}{n}$. ■

By the proof of Theorem 3.3.1 we can get a more general consequence.

Assume that $K(x)$ is a measurable function on $\mathbb{R}^n$. For a measurable function f on $\mathbb{R}^n$, let

$$Tf(x)=(K*f)(x).$$

For $1<r<\infty$, if there exists a constant $C>0$ such that for $\forall s>0$,

$$|\{x\in\mathbb{R}^n : |K(x)|>s\}|\le Cs^{-r},$$

then for $1\le p<r'$ and $\frac{1}{p}+\frac{1}{r}=\frac{1}{q}+1$,

(i) when $p=1$, T is weak type $(1,r)$;

(ii) when $1<p<r'$, T is (p,q)-type.

The following operator related to $T_{\Omega,\alpha}$ is a fractional maximal operator with homogeneous kernels. Its definition is

$$M_{\Omega,\alpha}f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|y|\le r}|\Omega(y)||f(x-y)|dy. \tag{3.3.12}$$

By the idea of the proof of (3.2.3), we can get a relation between $M_{\Omega,\alpha}$ and $T_{|\Omega|,\alpha}$.

Lemma 3.3.1 *Assume that* $0<\alpha<n$, $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ *satisfies (3.3.1). Then*

$$M_{\Omega,\alpha} \leq C(n,\alpha)T_{|\Omega|,\alpha}(|f|)(x).$$

Thus by Lemma 3.3.1 and Theorem 3.3.1, we can obtain the (L^p, L^q)-boundedness of fractional maximal operators with homogeneous kernels.

Theorem 3.3.2 *Assume that* $0<\alpha<n$, $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ *satisfies (3.3.1).*

(i) If $f \in L^1(\mathbb{R}^n)$*, then for* $\forall \lambda > 0$,

$$|\{x \in \mathbb{R}^n : M_{\Omega,\alpha}f(x) > \lambda\}| \leq \left(\frac{C}{\lambda}\|f\|_1\right)^{\frac{n}{n-\alpha}}.$$

(ii) If $f \in L^p(\mathbb{R}^n)(1<p\leq \frac{n}{\alpha})$*, then* $\|M_{\Omega,\alpha}f\|_q \leq C\|f\|_p$*, where* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ *and* $C=C(n,\alpha,p)$.

Proof. Omitted. ■

3.4 Weighted boundedness of $T_{\Omega,\alpha}$

In this section, we will extend the weighted boundedness of the Riesz potential I_α to the case of fractional integral operators with homogeneous kernels $T_{\Omega,\alpha}$. To the end, we will first show weighted boundedness of the fractional maximal operator $M_{\Omega,\alpha}$ with homogeneous kernels. In this section, we always assume that Ω is a homogeneous with degree zero (i.e. Ω satisfies (3.3.1)).

Theorem 3.4.1 *Suppose that* $0<\alpha<n, 1\leq s'<p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *If* $\Omega \in L^s(\mathbb{S}^{n-1})$ *and* $\omega(x)^{s'} \in A(\frac{p}{s'},\frac{q}{s'})$*, then there exists a constant* C *independent of* f *such that*

$$\left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha}f(x)\omega(x))^q\,dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.$$

Proof. From Hölder's inequality, it follows that

$$\begin{aligned}
M_{\Omega,\alpha}f(x) &= \sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|y|\le r}|\Omega(y)||f(x-y)|dy\\
&\le \sup_{r>0}\frac{1}{r^{n-\alpha}}\left(\int_{|y|\le r}|f(x-y)|^{s'}dy\right)^{\frac{1}{s'}}\left(\int_{|y|\le r}|\Omega(y)|^{s}dy\right)^{\frac{1}{s}}\\
&\le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\sup_{r>0}\left(\frac{1}{r^{n-\alpha s'}}\int_{|y|\le r}|f(x-y)|^{s'}dy\right)^{\frac{1}{s'}}\\
&= C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\left(M_{\alpha s}(|f|^{s'})(x)\right)^{\frac{1}{s'}}.
\end{aligned} \tag{3.4.1}$$

The hypothesis conditions imply that $0<\alpha s'<n, 1<\frac{p}{s'}<\frac{n}{\alpha s'}$ and $\frac{1}{(q/s')}=\frac{1}{(p/s')}-\frac{\alpha s'}{n}$. Applying Theorem 3.2.4 together with (3.4.1) yields that

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}(M_{\Omega,\alpha}f(x)\omega(x))^q dx\right)^{\frac{1}{q}} &\le C\left[\left(\int_{\mathbb{R}^n}(M_{\alpha s'}(|f|^{s'})(x)\omega(x)^{s'}(x))^{\frac{q}{s'}}dx\right)^{\frac{s'}{q}}\right]^{\frac{1}{s'}}\\
&\le C\left[\left(\int_{\mathbb{R}^n}(|f(x)|^{s'}\omega(x)^{s'}(x))^{\frac{p}{s'}}dx\right)^{\frac{s'}{p}}\right]^{\frac{1}{s'}}\\
&= C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

This finishes the proof of Theorem 3.4.1. ■

The next lemma gives a pointwise relationship between fractional integral operators with homogeneous kernels and fractional maximal operators with homogeneous kernels.

Lemma 3.4.1 *Suppose that $\varepsilon>0$ satisfies $0<\alpha-\varepsilon<\alpha+\varepsilon<n$, $x\in\mathbb{R}^n$. Then*

$$|T_{\Omega,\alpha}f(x)|\le C(n,\alpha,\varepsilon)(M_{\Omega,\alpha+\varepsilon}f(x))^{\frac{1}{2}}(M_{\Omega,\alpha-\varepsilon}f(x))^{\frac{1}{2}}. \tag{3.4.2}$$

Proof. Given $\varepsilon>0$ and $x\in\mathbb{R}^n$, take $\delta>0$ such that

$$\delta^{2\varepsilon}=\frac{M_{\Omega,\alpha+\varepsilon}f(x)}{M_{\Omega,\alpha-\varepsilon}f(x)}.$$

Set

$$T_{\Omega,\alpha}f(x) = \int_{|x-y|<\delta} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy + \int_{|x-y|\geq\delta} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy := \mathrm{I}_1 + \mathrm{I}_2.$$

Thus

$$\begin{aligned}|\mathrm{I}_1| &\leq \sum_{j=0}^{\infty} \int_{2^{-j-1}\delta\leq|x-y|<2^{-j}\delta} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|dy \\ &\leq \sum_{j=0}^{\infty} (2^{-j-1}\delta)^{-(n-\alpha)} \int_{|x-y|<2^{-j}\delta} |\Omega(x-y)||f(y)|dy \\ &\leq C\delta^{\varepsilon} M_{\Omega,\alpha-\varepsilon}f(x).\end{aligned}$$

Similarly,

$$\begin{aligned}|\mathrm{I}_2| &\leq \sum_{j=1}^{\infty} \int_{2^{j-1}\delta\leq|x-y|<2^{j}\delta} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|dy \\ &\leq C\sum_{j=1}^{\infty} (2^{j}\delta)^{-\varepsilon} \frac{1}{(2^j\delta)^{n-\alpha-\varepsilon}} \int_{|x-y|<2^{j}\delta} |\Omega(x-y)||f(y)|dy \\ &\leq C\delta^{-\varepsilon} M_{\Omega,\alpha+\varepsilon}f(x).\end{aligned}$$

Therefore

$$\begin{aligned}|T_{\Omega,\alpha}f(x)| &\leq C(\delta^{\varepsilon} M_{\Omega,\alpha-\varepsilon}f(x) + \delta^{-\varepsilon} M_{\Omega,\alpha+\varepsilon}f(x)) \\ &= 2C(M_{\Omega,\alpha+\varepsilon}f(x))^{\frac{1}{2}}(M_{\Omega,\alpha-\varepsilon}f(x))^{\frac{1}{2}}.\end{aligned}$$

■

To show the weighted boundedness of $T_{\Omega,\alpha}$, we also need the following property of $A(p,q)$.

Lemma 3.4.2 *Suppose that* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \omega \in A(p,q)$. *Then there exists* $\varepsilon > 0$ *such that*

(i) $\varepsilon < \alpha < \alpha + \varepsilon < n$;

(ii) $\frac{1}{p} > \frac{\alpha+\varepsilon}{n}, \frac{1}{q} < \frac{n-\varepsilon}{n}$;

(iii) $\omega \in A(p,q_\varepsilon), \omega \in A(p,\widetilde{q_\varepsilon})$ *hold, where*

$$\frac{1}{q_\varepsilon} = \frac{1}{p} - \frac{\alpha+\varepsilon}{n}, \quad \frac{1}{\widetilde{q_\varepsilon}} = \frac{1}{p} - \frac{\alpha-\varepsilon}{n}.$$

Proof. Since $\alpha > 0$ and $q > 1$, we may choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \alpha$ and

$$\frac{1}{q} + \frac{\varepsilon_1}{n} < 1.$$

Let

$$\frac{1}{q_{\varepsilon_1}} = \frac{1}{p} - \frac{\alpha - \varepsilon_1}{n} = \frac{1}{q} + \frac{\varepsilon_1}{n},$$

then $q > q_{\varepsilon_1} > 1$ and

$$1 + \frac{p'}{q} < 1 + \frac{p'}{q_{\varepsilon_1}}.$$

Applying Theorem 3.2.2 together with the property of A_p-weight lead to

$$\omega^{-p'} \in A_{1+\frac{p'}{q}} \subset A_{1+\frac{p'}{q_{\varepsilon_1}}}.$$

Applying Theorem 3.2.2 again, it is equivalent to

$$\omega \in A(p, q_{\varepsilon_1}). \tag{3.4.3}$$

On the other hand, by the property of A_p, there exits $\eta > 0$, such that $\eta < \frac{1}{q}$ and

$$\omega^{-p'} \in A_{1+p'(\frac{1}{q}-\eta)}.$$

Again choose $\varepsilon_2 > 0$, such that $\varepsilon_2 < \min\{\alpha, n-\alpha\}$, $\frac{1}{p} > \frac{\alpha+\varepsilon_2}{n}$ and $\frac{\varepsilon_2}{n} < \eta$. Let

$$\frac{1}{q_{\varepsilon_2}} = \frac{1}{p} - \frac{\alpha + \varepsilon_2}{n},$$

then $0 < \frac{1}{q_{\varepsilon_2}} < 1$ and $\frac{1}{q_{\varepsilon_2}} = \frac{1}{q} - \frac{\varepsilon_2}{n} > \frac{1}{q} - \eta$. By Theorem 3.2.2 and the property of A_p, we have

$$\omega(x)^{-p'} \in A_{1+p'(\frac{1}{q}-\eta)} \subset A_{1+\frac{p'}{q_{\varepsilon_2}}}.$$

The last is equivalent to

$$\omega \in A(p, q_{\varepsilon_2}). \tag{3.4.4}$$

Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then it is easy to prove that ε satisfies all the properties of ε_1 and ε_2. If we let $\frac{1}{q_\varepsilon} = \frac{1}{p} - \frac{\alpha+\varepsilon}{n}$, $\frac{1}{\widetilde{q_\varepsilon}} = \frac{1}{p} - \frac{\alpha-\varepsilon}{n}$, then by (3.4.3) and (3.4.4), we see that $\omega \in A(p, q_\varepsilon)$ and $\omega \in A(p, \widetilde{q_\varepsilon})$. ∎

Lemma 3.4.3 *Suppose that* $0 < \alpha < n, 1 \leq s' < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $\omega(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$. *Then there exists* $\varepsilon > 0$ *such that*

(i) $\varepsilon < \alpha < \alpha + \varepsilon < n$;

(ii) $\frac{1}{p} > \frac{\alpha+\varepsilon}{n}, \frac{1}{q} < \frac{n-\varepsilon}{n}$;

(iii) $\omega(x)^{s'} \in A(\frac{p}{s'}, \frac{q_\varepsilon}{s'}), \omega(x)^{s'} \in A(\frac{p}{s'}, \frac{\widetilde{q_\varepsilon}}{s'})$ *hold, where* q_ε *and* $\widetilde{q_\varepsilon}$ *are the same as in Lemma 3.4.2.*

Proof. This consequence follows directly from Lemma 3.4.2. ■

Next we will state and prove the weighted (L^p, L^q)-boundedness of $T_{\Omega,\alpha}$.

Theorem 3.4.2 *Suppose that* $0 < \alpha < n, 1 \leq s' < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $\Omega \in L^s(\mathbb{S}^{n-1})$ *and* $\omega(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$, *then there exists a constant* C *independent of* f *such that*

$$\left(\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)\omega(x)|^q dx\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.$$

Proof. Let $\varepsilon > 0$ be determined in Lemma 3.4.3. Let

$$l_1 = \frac{2q_\varepsilon}{q}, l_2 = \frac{2\widetilde{q_\varepsilon}}{q},$$

then $\frac{1}{l_1} + \frac{1}{l_2} = 1$. For the above ε, applying Lemma 3.4.1 and Hölder's inequality, we have that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)\omega(x)|^q dx\right)^{\frac{1}{q}} \\
\leq& C\left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha+\varepsilon} f(x)\omega(x))^{\frac{q}{2}} (M_{\Omega,\alpha-\varepsilon} f(x)\omega(x))^{\frac{q}{2}} dx\right)^{\frac{1}{q}} \\
\leq& C\left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha+\varepsilon} f(x)\omega(x))^{\frac{ql_1}{2}} dx\right)^{\frac{1}{ql_1}} \left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha-\varepsilon} f(x)\omega(x))^{\frac{ql_2}{2}} dx\right)^{\frac{1}{ql_2}} \\
=& C\left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha+\varepsilon} f(x)\omega(x))^{q_\varepsilon} dx\right)^{\frac{1}{2q_\varepsilon}} \left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha-\varepsilon} f(x)\omega(x))^{\widetilde{q_\varepsilon}} dx\right)^{\frac{1}{2\widetilde{q_\varepsilon}}}.
\end{aligned}$$

By Lemma 3.4.3 and Theorem 3.4.1, we have

$$\left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha+\varepsilon} f(x)\omega(x))^{q_\varepsilon} dx\right)^{\frac{1}{2q_\varepsilon}} \leq C\|f\|_{p,\omega^p}^{\frac{1}{2}}$$

and

$$\left(\int_{\mathbb{R}^n}(M_{\Omega,\alpha-\varepsilon}f(x)\omega(x))^{\widetilde{q}_\varepsilon}dx\right)^{\frac{1}{2\widetilde{q}_\varepsilon}} \le C\|f\|^{\frac{1}{2}}_{p,\omega^p}.$$

Therefore

$$\|T_{\Omega,\alpha}f\|_{q,\omega(x)^q} \le C\|f\|_{p,\omega^p}.$$

∎

Next we will state the dual form of Theorem 3.4.2.

Theorem 3.4.3 *Suppose that* $0<\alpha<n, 1<p<\frac{n}{\alpha}, \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ *and* $s>q$. *If* $\Omega\in L^s(\mathbb{S}^{n-1})$ *and* $\omega(x)^{-s'}\in A(\frac{q'}{s'},\frac{p'}{s'})$, *then there exists a constant* $C>0$ *independent of* f *such that*

$$\left(\int_{\mathbb{R}^n}|T_{\Omega,\alpha}f(x)\omega(x)|^q dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.$$

Proof. Let $\widetilde{\Omega}(x)=\overline{\Omega(-x)}$, then $\widetilde{\Omega}$ satisfies the conditions as Ω does. It is easy to check that $T_{\widetilde{\Omega},\alpha}$ is the dual operator of $T_{\Omega,\alpha}$. Hence

$$\begin{aligned}\|T_{\Omega,\alpha}f\|_{q,\omega(x)^q} &= \sup_{\|g\|_{q',\omega^{-q'}}\le 1}\left|\int_{\mathbb{R}^n}T_{\Omega,\alpha}f(x)g(x)dx\right| \\ &= \sup_{\|g\|_{q',\omega^{-q'}}\le 1}\left|\int_{\mathbb{R}^n}f(x)T_{\widetilde{\Omega},\alpha}g(x)dx\right| \\ &\le \|f\|_{p,\omega^p}\sup_{\|g\|_{q',\omega^{-q'}}\le 1}\|T_{\widetilde{\Omega},\alpha}g(x)\|_{p',\omega^{-p'}}.\end{aligned}$$

By the given condition, we have

$$\frac{1}{p'}=\frac{1}{q'}-\frac{\alpha}{n},\ \ s'<q'<\frac{n}{\alpha}\ \text{ and }\ (\omega^{-1})^{s'}\in A(\frac{q'}{s'},\frac{p'}{s'}).$$

From Theorem 3.4.2, it follows that

$$\|T_{\overline{\Omega},\alpha}g\|_{p',\omega^{-p'}} \le C\|g\|_{q',\omega^{-q'}}.$$

Thus

$$\|T_{\Omega,\alpha}f\|_{q,\omega^q} \le C\|f\|_{p,\omega^p}.$$

∎

As an immediate result of Theorem 3.4.3 and Lemma 3.3.1, we can get the dual form of Theorem 3.4.1:

Theorem 3.4.4 *Suppose that* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $s > q$. *If* $\Omega \in L^s(\mathbb{S}^{n-1})$ *and* $\omega(x)^{-s'} \in A(\frac{q'}{s'}, \frac{p'}{s'})$, *then there exists a constant* $C > 0$ *independent of* f *such that*

$$\left(\int_{\mathbb{R}^n} (M_{\Omega,\alpha} f(x)\omega(x))^q dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.$$

Note that in Theorem 3.4.2 and Theorem 3.4.4, the class of weight functions depend on s. The following theorem shows that this restriction can be dropped off.

Theorem 3.4.5 *Suppose that* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If for some* s *satisfying* $\frac{\alpha}{n} + \frac{1}{s} < \frac{1}{p} < \frac{1}{s'}$, $\Omega \in L^s(\mathbb{S}^{n-1})$, *furthermore, there exists* $1 < r < s/(\frac{n}{\alpha})'$ *such that* $\omega(x)^{r'} \in A(p,q)$, *then there exists a constant* C *independent of* f *such that*

$$\left(\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)\omega(x)|^q \, dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.$$

To prove Theorem 3.4.5, we need the following interpolation theorem with change of measure. This is a simple corollary of the Stein-Weiss Intepolation Theorem with change of measure (see [StW]).

Lemma 3.4.4 *Suppose that* $0 < \alpha < n, 1 < p_0 < p_1 < \frac{n}{\alpha}, \frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. *If a linear operator* T *satisfies*

$$\|Tf\|_{L^{q_0}(\omega_0^{q_0})} \le C_0 \|f\|_{L^{p_0}(\omega_0^{p_0})}$$

and

$$\|Tf\|_{L^{q_1}(\omega_1^{q_1})} \le C_1 \|f\|_{L^{p_1}(\omega_1^{p_1})},$$

then

$$\|Tf\|_{L^q(\omega^q)} \le C \|f\|_{L^p(\omega^p)},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},$$

$\omega = \omega_0^{1-\theta}\omega_1^{\theta}$ *and* $C \le C_0^{1-\theta} C_1^{\theta}$ $(0 < \theta < 1)$.

Proof. Omitted. ■

If we can show that, under the condition of Theorem 3.4.5, there exists a real number θ $(0<\theta<1)$, p_0, p_1, q_0, q_1, ω_0, ω_1 such that the following conditions hold:

$$1 \le s' < p_0 < p < p_1 < \frac{n}{\alpha}, \tag{3.4.5}$$

$$\frac{n}{n-\alpha} < q_0 < q < q_1 < s, \tag{3.4.6}$$

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n},\ \frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n},\ \frac{1}{p} = \frac{1-\theta}{p_0} - \frac{\theta}{p_1}, \tag{3.4.7}$$

$$\omega = \omega_0^{1-\theta}\omega_1^{\theta}, \tag{3.4.8}$$

$$\omega_0^{s'} \in A\left(\frac{p_0}{s'}, \frac{q_0}{s'}\right), \omega_1^{-s'} \in A\left(\frac{q_1}{s'}, \frac{p_1}{s'}\right), \tag{3.4.9}$$

then applying Theorem 3.4.2, Theorem 3.4.4 and Lemma 3.4.4, we can obtain the consequence of Theorem 3.4.5. Now we will validate the conditions of (3.4.5)-(3.4.9), respectively.

Let us recall a property on A_p weight: $\omega \in A_p$ if and only if there exist $u, v \in A_1$, such that $\omega(x) = u(x)v(x)^{1-p}$. A nonnegative locally integrable function $u(x)$ is called A_p weight, if there exists a constant $C>0$ such that for any square cube Q,

$$\frac{1}{|Q|}\int_Q u(y)dy \le Cu(x), \quad \text{a.e. } x \in Q.$$

Since $\omega^{r'} \in A(p,q)$, Theorem 3.2.2 (i) implies $\omega^{r'q} \in A_{q\frac{n-\alpha}{n}}$. Thus there exist $u, v \in A_1$, such that

$$\omega(x)^{r'q} = u(x)v(x)^{1-q\frac{n-\alpha}{n}}.$$

That is,

$$\omega(x) = u(x)^{\frac{1}{r'q}}v(x)^{\frac{1}{r'q}-\frac{n-\alpha}{nr'}}.$$

So write

$$\omega = (u^{\tau}v^{\beta})^{1-\theta}(u^{\gamma}v^{\delta})^{\theta}, \tag{3.4.10}$$

where τ, β, γ and δ satisfies

$$\tau(1-\theta)+\gamma\theta = \frac{1}{r'q}, \beta(1-\theta)+\delta\theta = \frac{1}{r'q} - \frac{n-\alpha}{r'n}. \tag{3.4.11}$$

Now write $\omega_0(x) = u(x)^\tau v(x)^\beta$ and $\omega_1(x) = u(x)^\gamma v(x)^\delta$. Notice that if $1 \le s' < p_0 < p < \frac{n}{\alpha}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, and set

$$\tau = \frac{1}{q_0}, \beta = -\frac{1}{s'(\frac{p_0}{s'})'},$$

then we have

$$\omega_0^{s'} \in A(\frac{p_0}{s'}, \frac{q_0}{s'}).$$

In fact, since $u, v \in A_1$, for any cube Q, we have that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q [\omega_0^{s'}(x)]^{\frac{q_0}{s'}}dx\right)^{\frac{s'}{q_0}} \left(\frac{1}{|Q|}\int_Q [\omega_0(x)^{s'}]^{-(\frac{p_0}{s'})'}dx\right)^{1/(\frac{p_0}{s'})'} \\
=&\left(\frac{1}{|Q|}\int_Q u(x)^{q_0\tau}v(x)^{q_0\beta}dx\right)^{\frac{s'}{q_0}} \left(\frac{1}{|Q|}\int_Q u(x)^{-\tau s'(\frac{p_0}{s'})'}v(x)^{-\beta s'(\frac{p_0}{s'})'}dx\right)^{1/(\frac{p_0}{s'})'} \\
\le& C\left(\frac{1}{|Q|}\int_Q v(x)dx\right)^{s'\beta} \left(\frac{1}{|Q|}\int_Q u(x)^{q_0\tau}dx\right)^{\frac{s'}{q_0}} \left(\frac{1}{|Q|}\int_Q u(x)dx\right)^{-s'\tau} \cdot \\
&\left(\frac{1}{|Q|}\int_Q v(x)^{-\beta s'(\frac{p_0}{s'})'}dx\right)^{1/(\frac{p_0}{s'})'} \\
\le& C.
\end{aligned}$$

Obviously, C is independent of θ. Applying the same method as the above, we can prove that if $\frac{n}{n-\alpha} < q < q_1 < s$ and $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$, then taking $\gamma = -\frac{1}{p_1'}, \delta = 1/s'(\frac{q_1'}{s'})'$, we have $\omega(x)^{-s'} \in A(\frac{q_1'}{s'}, \frac{p_1'}{s'})$.

Now let us calculate θ by (3.4.11). Note

$$\beta = -\left(s'(\frac{p_0}{s'})'\right)^{-1} = \frac{1}{p_0} - \frac{1}{s'}$$

and

$$\delta = \left(s'(\frac{q_1'}{s'})'\right)^{-1},$$

thus

$$\theta = \frac{\tau - \beta - \frac{n-\alpha}{r'n}}{\delta - \gamma - \beta + \tau} = \frac{\frac{1}{s'} - \frac{\alpha}{n} - \frac{n-\alpha}{r'n}}{2(\frac{1}{s'} - \frac{\alpha}{n})}. \tag{3.4.12}$$

Obviously $\theta < 1$.

Since $1 < r < s/(\frac{n}{\alpha})'$, there exists $\varepsilon > 0$ such that $\frac{1}{r} = \frac{n}{n-\alpha}(\frac{1}{s}+\varepsilon)$. Thus

$$\frac{1}{s'} - \frac{\alpha}{n} - \frac{n-\alpha}{r'n} = \frac{1}{s'} - \frac{\alpha}{n} - \frac{n-\alpha}{n}\left(1 - \frac{n}{n-\alpha}(\frac{1}{s}+\varepsilon)\right) = \varepsilon.$$

By (3.4.12), we have

$$\theta = \frac{\varepsilon}{2(\frac{1}{s'} - \frac{\alpha}{n})} > 0.$$

Thus we have explained that if θ was determined by (3.4.12), and (3.4.5)-(3.4.7) hold, then both (3.4.8) and (3.4.9) hold. Hence it remains to choose p_0, p_1, q_0, q_1 such that (3.4.5)-(3.4.7) hold.

Since $\frac{1}{p} > \frac{\alpha}{n} + \frac{1}{s}$ and $\theta > 0$, we have

$$\frac{1}{p(1-\theta)} - \frac{\alpha\theta}{n(1-\theta)} - \frac{\theta}{s(1-\theta)} > \frac{1}{p}. \tag{3.4.13}$$

By (3.4.13) and the fact that $\frac{1}{p} < \frac{1}{s'}$, we choose p_0 satisfies

$$\frac{1}{p} < \frac{1}{p_0} < \min\{\frac{1}{s'}, \frac{1}{p(1-\theta)} - \frac{\alpha\theta}{n(1-\theta)} - \frac{\theta}{s(1-\theta)}\}. \tag{3.4.14}$$

From this it follows that $s' < p_0 < p$ and $\frac{1}{p} > \frac{1-\theta}{p_0} + \frac{\alpha\theta}{n}$.

Choose $\sigma > 0$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \left(\frac{\alpha}{n} + \sigma\right)\theta. \tag{3.4.15}$$

Now let $\frac{1}{p_1} = \frac{\alpha}{n} + \sigma$. Then by $p_1 < \frac{n}{\alpha}$ and $p > p_0$ we see that $s' < p_0 < p < p_1 < \frac{n}{\alpha}$. So (3.4.5) holds. By (3.4.15) we see that (3.4.7) also holds. Let $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}, \frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$, then it is obvious that

$$\frac{n}{n-\alpha} < q_0 < q < q_1.$$

Finally we will show $q_1 < s$. By (3.4.14) we can get

$$\frac{1}{p\theta} - \frac{1-\theta}{p_0\theta} - \frac{\alpha}{n} > \frac{1}{s}.$$

This is equivalent to

$$\frac{1}{p_1} - \frac{\alpha}{n} > \frac{1}{s}, \text{ i.e. } \frac{1}{q_1} > \frac{1}{s}.$$

Thus (3.4.6) holds. This finishes the proof of Theorem 3.4.5.

3.5 Commutators of Riesz potential

In this section we will study (L^p, L^q)-boundedness of commutators of the Riesz potential I_α. We will also illustrate that boundedness of commutators of I_α can characterize $\mathrm{BMO}(\mathbb{R}^n)$ spaces. First we will give some definitions and related results.

Suppose that $b \in L_{loc}(\mathbb{R}^n)$, then the commutator generated by b and the Riesz potential I_α is defined by

$$[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x) = \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]}{|x-y|^{n-\alpha}} f(y)dy. \quad (3.5.1)$$

Let us first formulate the following consequences.

Theorem 3.5.1 *Suppose that $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

To prove the theorem, we will prove the following lemma first.

Lemma 3.5.1 *Suppose that $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $1 < r, s < p$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, then there exists a constant C independent of b such that, for any $f \in L^p(\mathbb{R}^n)$,*

$$M^\sharp\big([b, I_\alpha]f\big)(x) \le C\|b\|_{\mathrm{BMO}} \left[\big(M(|I_\alpha f|^r)(x)\big)^{\frac{1}{r}} + \big(M_{\alpha s}(|f|^s)(x)\big)^{\frac{1}{s}}\right], \quad a.e.\ x \in \mathbb{R}^n. \quad (3.5.2)$$

Proof. Fix a cube Q and set

$$\begin{aligned}[b, I_\alpha]f(x) &= (b(x)-b_Q)I_\alpha f(x) - I_\alpha((b-b_Q)f\chi_{2Q})(x) - I_\alpha((b-b_Q)f\chi_{(2Q)^c})(x) \\ &:= a_1(x) - a_2(x) - a_3(x).\end{aligned}$$

Applying Hölder's inequality and (2.4.2), we have that

$$\begin{aligned}\frac{1}{|Q|}\int_Q |a_1(x)|dx &\le \left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|^{r'}dx\right)^{\frac{1}{r'}} \left(\frac{1}{|Q|}\int_Q |I_\alpha f(x)|^r dx\right)^{\frac{1}{r}} \\ &\le C\|b\|_{\mathrm{BMO}}\big(M(|I_\alpha f|^r)\big)^{\frac{1}{r}}.\end{aligned} \quad (3.5.3)$$

Since $1 < s < p$, we can choose $\gamma > 1, \beta > 1$ such that $\gamma\beta = s$. Thus it implies $1 < \beta < p < \frac{n}{\alpha}$. Then there exists $u > \beta$ such that $\frac{1}{u} = \frac{1}{\beta} - \frac{\alpha}{n}$. Applying the Hardy-Littlewood-Sobolev theorem (Theorem 3.1.1), we have that

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |a_2(x)|dx &\le \left(\frac{1}{|Q|}\int_Q |I_\alpha((b-b_Q)f\chi_{2Q})(x)|^u dx\right)^{\frac{1}{u}} \\
&\le C\frac{1}{|Q|^{\frac{1}{u}}}\left(\int_{2Q} |b(x)-b_Q|^\beta |f(x)|^\beta dx\right)^{\frac{1}{\beta}} \\
&\le C\frac{1}{|Q|^{\frac{1}{u}}}\left(\int_{2Q} |b(x)-b_Q|^{\gamma'\beta} dx\right)^{\frac{1}{\beta\gamma'}}\left(\int_{2Q} |f(x)|^{\gamma\beta} dx\right)^{\frac{1}{\gamma\beta}} \\
&\le C\|b\|_{\mathrm{BMO}}|Q|^{\frac{\alpha}{n}}\left(\frac{1}{|Q|}\int_{2Q} |f(x)|^s dx\right)^{\frac{1}{s}} \\
&= C\|b\|_{\mathrm{BMO}}\left(\frac{1}{|Q|^{1-\frac{\alpha s}{n}}}\int_{2Q} |f(x)|^s dx\right)^{\frac{1}{s}} \\
&\le C\|b\|_{\mathrm{BMO}}\left(M_{\alpha s}(|f|^s)(x)\right)^{\frac{1}{s}}.
\end{aligned} \tag{3.5.4}$$

Denote x_0 as the center of Q, then for $x \in Q$ and $y \in (2Q)^c$, we have the estimate: $|x-y| \sim |x_0-y|$. Note that

$$\begin{aligned}
&|I_\alpha((b-b_Q)f\chi_{(2Q)^c})(x) - I_\alpha((b-b_Q)f\chi_{(2Q)^c})(x_0)| \\
&\le \int_{\mathbb{R}^n\setminus 2Q}\left|\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}}\right||b(y)-b_Q||f(y)|dy \\
&\le C\int_{\mathbb{R}^n\setminus 2Q}\frac{|x-x_0|}{|x_0-y|^{n-\alpha+1}}|b(y)-b_Q||f(y)|dy \\
&\le C\left(\int_{\mathbb{R}^n\setminus 2Q}\frac{|x-x_0|}{|x_0-y|^{n+1}}|b(y)-b_Q|^{s'}dy\right)^{\frac{1}{s'}} \\
&\quad\times\left(\int_{\mathbb{R}^n\setminus 2Q}\frac{|x-x_0|}{|x_0-y|^{n+1-\alpha s}}|f(y)|^s dy\right)^{\frac{1}{s}} \\
&\le C\|b\|_{\mathrm{BMO}}\left(M_{\alpha s}(|f|^s)(x)\right)^{\frac{1}{s}}.
\end{aligned} \tag{3.5.5}$$

From (3.5.3)-(3.5.5) it follows that

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|[b,I_\alpha](f)(x) - I_\alpha((b-b_Q)f\chi_{(2Q)^c})(x_0)\right| dx \\
&\le \frac{1}{|Q|}\int_Q |a_1(x)|dx + \frac{1}{|Q|}\int_Q |a_2(x)|dx \\
&\quad + \frac{1}{|Q|}\int_Q |a_3(x) - I_\alpha((b-b_Q)f\chi_{(2Q)^c})(x_0)|dx \\
&\le C\|b\|_{\mathrm{BMO}}\left[\left(M(|I_\alpha f|^r)(x)\right)^{\frac{1}{r}} + \left(M_{\alpha s}(|f|^s)(x)\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Note that C is independent of Q, thus (3.5.2) holds. ■

Let us now return to the proof of Theorem 3.5.1. First suppose $b \in \mathrm{BMO}(\mathbb{R}^n)$, by Lemma 2.4.2 and Lemma 3.5.1, we have that

$$\begin{aligned}
\|[b,I_\alpha]f\|_q &\le \|M([b,I_\alpha]f)\|_q \\
&\le C(n,q)\|M^\sharp([b,I_\alpha]f)\|_q \\
&\le C\|b\|_{\mathrm{BMO}}\left(\|(M(|I_\alpha f|^r))^{\frac{1}{r}}\|_q + \|(M_{\alpha s}(|f|^s))^{\frac{1}{s}}\|_q\right).
\end{aligned} \tag{3.5.6}$$

Note that $1 < r < p < q$. Since M is of $(\frac{q}{r}, \frac{q}{r})$-type, by Theorem 3.1.1, we have

$$\left\|(M(|I_\alpha f|^r))^{\frac{1}{r}}\right\|_q \le C\,\|(I_\alpha f)^r\|_{\frac{q}{r}}^{\frac{1}{r}} \le C\|f\|_p.$$

As long as we choose $s > 1$ such that $0 < \alpha < \alpha s < n$, then we have $1 < \frac{p}{s} < \frac{n}{\alpha s}$ and $\frac{1}{q/s} = \frac{1}{p/s} - \frac{\alpha s}{n}$. Applying Theorem 3.2.1 yields that

$$\left\|(M_{\alpha s}(|f|^s))^{\frac{1}{s}}\right\|_q \le C\|f\|_p.$$

Thus it follows from (3.5.6) that

$$\|[b,I_\alpha]f\|_q \le C\|b\|_{\mathrm{BMO}}\|f\|_p.$$

Next we will give the proof of the necessity. Choose $z_0 \in \mathbb{R}^n, \delta > 0$ such that in the neighborhood $\{z : |z - z_0| < \delta\sqrt{n}\ \}$, function $|z|^{n-\alpha}$ can be represented as a Fourier series which absolutely converges. That is

$$|z|^{n-\alpha} = \sum_{m=0}^{\infty} a_m \mathrm{e}^{i<v_m,z>}.$$

Let $z_1 = \frac{z_0}{\delta}$. For any cube $Q = Q(x_0, r)$, let $y_0 = x_0 - rz_1, Q' = Q'(y_0, r)$. Then for $x \in Q, y \in Q'$, we have that

$$\left|\frac{x-y}{r} - z_1\right| \leq \left|\frac{x-x_0}{r}\right| + \left|\frac{y-y_0}{r}\right| \leq \sqrt{n}.$$

Now set $s(x) = \mathrm{sgn}[b(x) - b_{Q'}]$, then

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}|dx \\
&= \int_Q (b(x) - b_{Q'})s(x)dx \\
&= |Q'|^{-1}\int_Q\int_{Q'}(b(x) - b(y))s(x)dydx \\
&= \delta^{\alpha-n}r^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}\left|\frac{\delta(x-y)}{r}\right|^{n-\alpha} s(x)\chi_Q(x)\chi_{Q'}(y)dydx \\
&= Cr^{-\alpha}\sum_m a_m\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}\mathrm{e}^{i<v_m,\frac{\delta}{r}(x-y)>}s(x)\chi_Q(x)\chi_{Q'}(y)dydx.
\end{aligned}$$

Set

$$f_m(y) = \mathrm{e}^{-i\frac{\delta}{r}<v_m,y>}\chi_{Q'}(y)$$

and

$$g_m(x) = \mathrm{e}^{i\frac{\delta}{r}<v_m,x>}s(x)\chi_Q(x),$$

then $f_m \in L^p(\mathbb{R}^n)$ and

$$\begin{aligned}
\int_Q |b(x) - b_{Q'}|dx &\leq Cr^{-\alpha}\sum_m a_m\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}f_m(y)dyg_m(x)dx \\
&\leq Cr^{-\alpha}\sum_m |a_m|\int_{\mathbb{R}^n}|[b,I_\alpha](f_m)(x)||g_m(x)|dx \\
&\leq Cr^{-\alpha}\sum_m |a_m|\int_Q |[b,I_\alpha](f_m)(x)|dx.
\end{aligned}$$

Applying Hölder's inequality and (L^p, L^q)-boundedness of $[b, I_\alpha]$, we conclude that

$$\begin{aligned}
\int_Q |b(x) - b_{Q'}|dx &\leq Cr^{-\alpha}\sum_m |a_m||Q|^{\frac{1}{q'}}\left(\int_Q |[b,I_\alpha](f_m)(x)|^q dx\right)^{\frac{1}{q}} \\
&\leq Cr^{-\alpha}\sum_m |a_m||Q|^{\frac{1}{q'}}|Q'|^{\frac{1}{p}} \\
&\leq C|Q|.
\end{aligned}$$

This is equivalent to $b \in \mathrm{BMO}(\mathbb{R}^n)$, and therefore the proof of Theorem 3.5.1 is finished.

For the commutator $[b, M_\alpha]$ of fractional maximal operator M_α, there are some results parallel to Theorem 3.5.1. Here $[b, M_\alpha]$ is defined by

$$[b, M_\alpha](f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y-x|\le r} |b(x) - b(y)||f(y)|dy.$$

Theorem 3.5.2 *Suppose that* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then commutator* $[b, M_\alpha]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *if and only if* $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Proof. Applying the same method in the proof of (3.2.3), we obtain that

$$[b, M_\alpha](f)(x) \le \gamma(\alpha)[b, I_\alpha](|f|)(x). \quad \forall x \in \mathbb{R}^n. \tag{3.5.7}$$

Thus when $b \in \mathrm{BMO}(\mathbb{R}^n)$, by (3.5.7) and Theorem 3.5.1 we deduce that $[b, M_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

On the other hand, suppose that $[b, M_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Choose any cube Q in $\mathbb{R}^n$,

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |b(x) - b_Q|dx &\le \frac{1}{|Q|^2}\int_Q\int_Q |b(x) - b(y)|dydx \\
&= \frac{1}{|Q|^{1+\frac{\alpha}{n}}}\int_Q \frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q |b(x) - b(y)|\chi_Q(y)dydx \\
&\le \frac{1}{|Q|^{1+\frac{\alpha}{n}}}\int_Q [b, M_\alpha](\chi_Q)(x)dx \\
&\le \frac{1}{|Q|^{1+\frac{\alpha}{n}}}|Q|^{\frac{1}{q'}}\left(\int_Q ([b, M_\alpha](\chi_Q)(x))^q dx\right)^{\frac{1}{q}} \\
&\le C\frac{1}{|Q|^{1+\frac{\alpha}{n}}}|Q|^{\frac{1}{q'}}|Q|^{\frac{1}{p}} = C.
\end{aligned}$$

Thus $b \in \mathrm{BMO}(\mathbb{R}^n)$. ∎

3.6 Commutators of fractional integrals with rough kernels

In this section we will discuss the weighted (L^p, L^q)-boundedness of commutators generated by fractional integral operators with rough kernels

$T_{\Omega,\alpha}$ and BMO functions. First we will give their definitions. Suppose $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then the commutator generated by $T_{\Omega,\alpha}$ and b is defined by

$$\begin{aligned}[b, T_{\Omega,\alpha}](f)(x) &= b(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(bf)(x)\\ &= \int_{\mathbb{R}^n} \frac{[b(x)-b(y)]}{|x-y|^{n-\alpha}}\Omega(x-y)f(y)dy.\end{aligned}$$

The commutator $[b, M_{\Omega,\alpha}]$ generated by $M_{\Omega,\alpha}$ and b is defined by

$$[b, M_{\Omega,\alpha}](f)(x) = \sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|\le r}|b(x)-b(y)||\Omega(x-y)||f(y)|dy.$$

We begin with indicating some relations between $A(p,q)$-weight and $\mathrm{BMO}(\mathbb{R}^n)$.

Lemma 3.6.1 *Suppose that $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $\omega \in A(p,q)$, then there exists $\varepsilon > 0$, such that $\omega^{1+\varepsilon} \in A(p,q)$.*

Proof. Applying Theorem 3.2.2, we have $\omega(x)^q \in A_{\frac{q(n-\alpha)}{n}}$. By the property of A_p, there exists $\varepsilon > 0$ such that $\omega^{q(1+\varepsilon)} \in A_{\frac{q(n-\alpha)}{n}}$. Then we can get the desired consequence by applying Theorem 3.2.2 again. ■

Lemma 3.6.2 *Suppose that $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then there exists $\lambda > 0$ such that $e^{\lambda b} \in A(p,q)$.*

Proof. By the John-Nirenberg inequality and the inverse Hölder's inequality of A_p, there exists a $\lambda_0 > 0$, such that $e^{\lambda_0 b} \in A_{\frac{q(n-\alpha)}{n}}$ (see Garciá-Cuerva and Rubio de Francia [GaR]). Now take $\lambda = \frac{\lambda_0}{q}$, then $\mathrm{e}^{\lambda b q} \in A_{\frac{q(n-\alpha)}{n}}$. By Theorem 3.2.2, this leads to $\mathrm{e}^{\lambda b} \in A(p,q)$. ■

Lemma 3.6.3 *Suppose that $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For $\lambda > 0$, there exists a $\eta > 0$ such that if $b \in \mathrm{BMO}$ and $\|b\|_* < \eta$, then $e^{\lambda b} \in A(p,q)$.*

Proof. It is not difficult to show that if we set

$$\eta = \min\left\{\frac{C}{\lambda q}, \frac{C[q(n-\alpha)/n-1]}{\lambda q}\right\},$$

here C is the absolute constant in the John-Nirenberg inequality, then when $\|b\|_{\mathrm{BMO}} < \eta$ we have (see [GaR])

$$\mathrm{e}^{\lambda qb} \in A_{\frac{q(n-\alpha)}{n}}.$$

By Theorem 3.2.2, this is equivalent to $\mathrm{e}^{\lambda b(x)} \in A(p,q)$. ■

Now let us turn our attention to the weighted (L^p, L^q)-boundedness of $[b, T_{\Omega,\alpha}]$.

Theorem 3.6.1 *Suppose that* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Let* $\Omega \in L^s(\mathbb{S}^{n-1}), s > 1$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *If* p, q, s, ω *satisfy one of the following three conditions, then* $\|[b, T_{\Omega,\alpha}]f\|_{q,\omega^q} \le C\|f\|_{p,\omega^p}$*:*

(i) $s' < p$ *and* $\omega(x)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$*;*

(ii) $q < s$ *and* $\omega^{-s'} \in A(\frac{q'}{s'}, \frac{p'}{s'})$*;*

(iii) $\frac{\alpha}{n} + \frac{1}{s} < \frac{1}{p} < \frac{1}{s'}, 1 < r < s/(\frac{n}{\alpha})'$ *and* $\omega(x)^{r'} \in A(p,q)$.

Proof. First we will give the proof of (i). By (2.4.29), we obtain that

$$\begin{aligned}
|[b, T_{\Omega,\alpha}]f(x)| &= \left|\int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}}\Omega(x-y)f(y)dy\right| \\
&= \left|\int_{\mathbb{R}^n} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{e}^{i\theta}[b(x)-b(y)]}\mathrm{e}^{-i\theta}d\theta \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\mathrm{e}^{-b(y)\cos\theta}dy \cdot \mathrm{e}^{b(x)\cos\theta}d\theta.
\end{aligned}$$

Let $g_\theta(y) = f(x)\mathrm{e}^{-b(y)\cos\theta}, \theta \in [0, 2\pi]$. Since $f \in L^p(\omega^p)$, we have, for any $\theta \in [0, 2\pi]$, that

$$g_\theta \in L^p(\omega^p \mathrm{e}^{pb\cos\theta}) \quad \text{and} \quad \|g_\theta\|_{p,\omega^p\mathrm{e}^{pb\cos\theta}} = \|f\|_{p,\omega^p} \tag{3.6.1}$$

Thus

$$|[b, T_{\Omega,\alpha}]f(x)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|T_{|\Omega|}(|g_\theta|)(x)\right| \mathrm{e}^{b(x)\cos\theta}d\theta.$$

From the Minkowski's inequality, it follows that

$$\|[b, T_{\Omega,\alpha}]f\|_{q,\omega^q} \le \frac{1}{2\pi}\int_0^{2\pi} \|T_{|\Omega|,\alpha}(|g_\theta|)\|_{q,\omega^q\mathrm{e}^{qb\cos\theta}}d\theta. \tag{3.6.2}$$

By the given conditions, $1 < \frac{1}{(p/s')} < \frac{n}{\alpha s'}, 0 < \alpha s' < n$ and $\frac{1}{(q/s')} = \frac{1}{(p/s')} - \frac{\alpha s'}{n}$ together with Lemma 3.6.1, we see that there exists $\varepsilon > 0$ such that

$\omega^{s'(1+\varepsilon)} \in A(\frac{p}{s'}, \frac{q}{s'})$. Then by Theorem 3.4.1, for any $\varphi \in L^p(\omega^{p(1+\varepsilon)})$, we have that

$$\|T_{|\Omega|,\alpha}(\varphi)\|_{q,\omega^{q(1+\varepsilon)}} \leq C_1\|\varphi\|_{p,\omega^{p(1+\varepsilon)}}. \tag{3.6.3}$$

On the other hand, if we choose $\lambda = \frac{s'(1+\varepsilon)}{\varepsilon}$, then by Lemma 3.6.3, there exists $\eta > 0$ such that, for $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|b\|_{\mathrm{BMO}} < \eta$, $\mathrm{e}^{\lambda b} \in A(\frac{p}{s'}, \frac{q}{s'})$, i.e. $\left(\mathrm{e}^{(1+\varepsilon)b/\varepsilon}\right)^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$.

Notice that if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and for all $|t| \leq 1$ then $tb \in \mathrm{BMO}$ and $\|tb\|_{\mathrm{BMO}} \leq \|b\|_{\mathrm{BMO}}$. Thus for all $|t| \leq 1$, we have

$$\left(\mathrm{e}^{tb(1+\varepsilon)/\varepsilon}\right)^{s'} \in A\left(\frac{p}{s'}, \frac{q}{s'}\right). \tag{3.6.4}$$

Thus, without loss of generality, we can prove Theorem 3.6.1 (i) only in the case $\|b\|_{\mathrm{BMO}} < \eta$. Applying Theorem 3.4.1 again and (3.6.4), for any $\theta \in [0, 2\pi]$ and $\varphi \in L^p(\mathrm{e}^{pb(1+\varepsilon)\cos\theta/\varepsilon})$, we have that

$$\|T_{|\Omega|,\alpha}(\varphi)\|_{q,\mathrm{e}^{qb(1+\varepsilon)\cos\theta/\varepsilon}} \leq C_2\|\varphi\|_{p,\mathrm{e}^{pb(1+\varepsilon)\cos\theta}/\varepsilon}. \tag{3.6.5}$$

Here C_2 is independent of θ.

By (3.6.3) and (3.6.5), applying the Stein-Weiss Intepolation Theorem with change measure (Lemma 3.4.4), for any $\theta \in [0, 2\pi]$ and $\varphi \in L^p(\omega^p\mathrm{e}^{pb\cos\theta})$, we obtain that

$$\|T_{|\Omega|,\alpha}(\varphi)\|_{q,\omega^q\mathrm{e}^{qb\cos\theta}} \leq C\|\varphi\|_{p,\omega^p\mathrm{e}^{pb\cos\theta}}, \tag{3.6.6}$$

where $C \leq C_1^{\frac{1}{1+\varepsilon}} C_2^{\frac{\varepsilon}{1+\varepsilon}}$ and C is independent of θ, φ.

(3.6.1), (3.6.2) and (3.6.6) yield that

$$\|[b, T_{\Omega,\alpha}](f)\|_{q,\omega^q} \leq \frac{1}{2\pi}\int_0^{2\pi} C\|g_\theta\|_{p,\omega^{pb\cos\theta}}d\theta = C\|f\|_{p,\omega^p}.$$

This finishes the proof of consequence (i).

Using the consequence (i) and the method in the proof of Theorem 3.4.3, we can easily get the consequence (ii). Note that if we set $\widetilde{\Omega}(x) = -\overline{\Omega(-x)}$, then $[b, T_{\widetilde{\Omega},\alpha}]$ is the dual operator of $[b, T_{\Omega,\alpha}]$.

By the consequence (i)(ii) as well as the interpolation theorem with change of measure (Lemma 3.4.4) and using the method in the proof of Theorem 3.4.5, we can get the proof of the consequence (iii), therefore we omit the details here. ∎

Remark 3.6.1 *Let $m \in \mathbb{N}$. The commutator of degree m generated by $T_{\Omega,\alpha}$ and b is defined by*

$$[b, T_{\Omega,\alpha}]^m(f)(x) = [b, \cdots, [b, T_{\Omega,\alpha}]] = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x-y|^{n-\alpha}} \Omega(x-y) f(y) dy.$$

Using the method in the proof of Theorem 3.6.1 and the mathematical induction, we can prove that, under the conditions of Theorem 3.6.1, the consequences of weighted boundedness hold for $[b, T_{\Omega,\alpha}]^m$.

Remark 3.6.2 *Using the method in the proof of (3.2.3) we have that*

$$[b, M_{\Omega,\alpha}]f(x) \leq \gamma(\alpha)[b, T_{|\Omega|,\alpha}](|f|)(x), \quad \forall x \in \mathbb{R}^n. \tag{3.6.7}$$

By (3.6.7) we see that under the conditions of Theorem 3.6.1, the consequences of weighted boundedness still hold for $[b, M_{\Omega,\alpha}]$. If $m \in \mathbb{N}$, and the commutator of degree m generated by $M_{\Omega,\alpha}$ and b is defined by

$$[b, M_{\Omega,\alpha}]^m(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|\leq r} |b(x) - b(y)|^m |\Omega(x-y)||f(y)| dy.$$

Similarly, it is easy to prove that under the conditions of Theorem 3.6.1, the consequences of weighted boundedness hold for $[b, M_{\Omega,\alpha}]^m$.

3.7 Notes and references

Lemma 3.1.1 was proved by Hedberg [He]. The formula (3.2.3) shows that the fractional maximal operator can be dominated by the Riesz potential in pointwise sense, which is a special case of Lemma 3.3.1(i.e. $\Omega \equiv 1$). And Lemma 3.3.1 was proved by Ding [Di3].

The definition of $A(p, q)$ weights was first introduced by Muckenhoupt and Wheeden [MuW2]. Theorem 3.2.3, Theorem 3.2.4 and Theorem 3.2.5 come from [MuW2]. The fractional integral operator with homogeneous kernel was first introduced by Muckenhoupt and Wheeden [MuW1].In this paper, they obtained the power weighted (L^p, L^q)-boundedness of $T_{\Omega,\alpha}$ $(1 < p < \frac{n}{\alpha})$. The theorems presented in this section are generalizations of their consequences. As a complement of Muckenhoupt and Wheeden's consequences mentioned above, Ding [Di3] gave the power weighted weak $(1, \frac{n}{n-\alpha})$-boundedness of $T_{\Omega,\alpha}$.

In 1988, Adams (see [Ad]) proved that if $0 < \alpha < n$ and $f \in L^{\frac{n}{\alpha}}(B)$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$, then

$$\frac{1}{|B|}\int_B \exp\left\{ n \left| \frac{I_\alpha(f)(x)}{\|f\|_{\frac{n}{\alpha}}} \right|^{\frac{n}{n-\alpha}} \right\} dx \leq C,$$

where $C = C(\alpha, n)$. This result can be viewed as a replacement for the boundedness of I_α when $p = \frac{n}{\alpha}$. In 1996, Ding and Lu [DiL1] obtained a similar result for fractional integral operators with rough kernels.

All theorems in Section 4 of this chapter are taken from Ding and Lu [DiL3]. And Lemma 3.4.1 is also taken from [DiL3]. Its idea of the proof comes from Welland [We].

The necessity of Theorem 3.5.1 with $n - \alpha$ being an even and the sufficiency of the theorem were first proved by Chanillo [Cha] in 1982. The proof given here is simpler which comes from Ding [Di2].

There are counterexamples showing that when $b \in \mathrm{BMO}$, the commutator $[b, I_\alpha]$ is not $(L^1, L^{\frac{n}{n-\alpha},\infty})$-type. However Ding, Lu and Zhang [DiLZ], as well as Cruz-Uribe and Fiorenza [CrF] independently, proved that when $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $p = 1$, the commutator $[b, I_\alpha]$ satisfies an estimate of weak $L\log^+ L$ type.

Theorem 3.6.1 was proved by Ding [Di1]. In 1999, Ding, Lu [DiL4] established weighted norm inequalities for commutators generated by fractional integral operators with rough kernels and BMO functions. A two-weight weak-type norm inequality for the commutator generated by the Riesz potential and BMO function was obtained by Liu and Lu [LiL].

As space is limited, we do not mention the boundedness of homogeneous fractional integrals on Hardy spaces in this chapter. For related results, we refer to Ding, Lu [DiL5] and Lu, Wu [LuWu]. In [LuWu], the authors established some equivalent characterizations for $(H^1, L^{n/(n-\alpha)})$ type boundedness of commutators of fractional integrals.

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Chapter 4

Oscillatory Singular Integrals

Oscillatory integrals have been an essential part of harmonic analysis. Many important operators in harmonic analysis are some versions of oscillatory integrals, such as the Fourier transform, Bochner-Riesz means, Radon transform and so on. However, the object we study in this chapter is a class of oscillatory singular integrals with polynomial phases which is closely related to the Radon transform.

4.1 Oscillatory singular integrals with homogeneous smooth kernels

Suppose that K is a homogeneous Calderón-Zygmund kernel in $\mathbb{R}^n$. Precisely, K satisfies the following conditions:

(i) K is a C^1 function away from the origin,

(ii) K is homogeneous of degree $-n$, and

(iii) the mean-value of K on the unit sphere vanishes.

Let $P(x, y)$ be a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. Consider an operator T of the following form

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,\, y)} K(x - y) f(y) dy. \tag{4.1.1}$$

Let us first begin with L^p-boundedness of T in this section.

Theorem 4.1.1 *Suppose that K satisfies (i), (ii) and (iii). Then T defined by (4.1.1) is bounded on $L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and bound depending only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.*

In order to prove Theorem 4.1.1, we need to establish some lemmas on some general inequalities for polynomials in $\mathbb{R}^n$.

Lemma 4.1.1 *Suppose that $P(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha$ is a polynomial of degree d, and $\varepsilon < 1/d$. Then*

$$\sup_{y\in\mathbb{R}^n} \int_{|x|\le 1} |P(x-y)|^{-\varepsilon} dx \le A_\varepsilon \left(\sum_{|\alpha|=d} |a_\alpha| \right)^{\varepsilon},$$

where bound A_ε depends only on ε (and the dimension n), but not on the coefficients a_α.

Lemma 4.1.2 *Suppose that $P(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha$ is a homogeneous polynomial of degree d in $\mathbb{R}^n$ and $\varepsilon < 1/d$. Then*

$$\int_{\mathbb{S}^{n-1}} |P(x)|^{-\varepsilon} d\sigma(x) \le A_\varepsilon \left(\sum_{|\alpha|=d} |a_\alpha| \right)^{\varepsilon},$$

which bound A_ε depends only on ε (and the dimension n), but not on the coefficients a_α.

Lemma 4.1.3 (van der Corput) *Suppose that a real-valued $\phi \in C^k[a,b]$, $k > 1$, and $|\phi^{(k)}(t)| \ge 1$ for all $t \in (a,b)$. Then*

$$\left| \int_a^b e^{i\lambda\phi(t)} dt \right| \le C_k \lambda^{-1/k},$$

where $\lambda \in \mathbb{R}$, and C_k is independent of a, b and ϕ.

Let us consider an operator $\mathcal{T}$ of more general form

$$\mathcal{T}f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x,y) f(y) dy, \tag{4.1.2}$$

where K is a distribution, and for $x \ne y$, that is a function satisfying

$$|K(x,y)| \le \frac{A}{|x-y|^n}, \quad x \ne y. \tag{4.1.3}$$

For every $\varepsilon > 0$, consider the truncated operator $\mathcal{T}_\varepsilon$ defined by

$$\mathcal{T}_\varepsilon f(x) = \int_{|x-y|<\varepsilon} K(x,y)f(y)dy. \tag{4.1.4}$$

Lemma 4.1.4 *Fix p, $1 \le p < \infty$. If $\mathcal{T}$ is bounded on $L^p(\mathbb{R}^n)$, then so is each $\mathcal{T}_\varepsilon$. Moreover if $\|\cdot\|$ denotes the operator norm in L^p, we have the estimate*

$$\|\mathcal{T}_\varepsilon\| \le C\{\|\mathcal{T}\| + A\}$$

with A arising in (4.1.3) and C independent of K.

Proof. For any fixed $h \in \mathbb{R}^n$, we split f into three parts as

$$f(y) = \begin{cases} f_1(y), & |y-h| < \varepsilon/2, \\ f_2(y), & \varepsilon/2 \le |y-h| < 5\,\varepsilon/4, \\ f_3(y), & \text{otherwise.} \end{cases}$$

It is easy to see that if $|x-h| < \varepsilon/4$, then

$$\mathcal{T}_\varepsilon f_1(x) = \mathcal{T} f_1(x).$$

So by our assumption on K, we have that

$$\begin{aligned}\int_{|x-h|<\varepsilon/4} |\mathcal{T}_\varepsilon f_1(x)|^p dx =& \int_{|x-h|<\varepsilon/4} \left| \int_{\mathbb{R}^n} K(x,y) f_1(y)dy \right|^p dx \\ \le& \int_{\mathbb{R}^n} |\mathcal{T} f_1(x)|^p dx \\ \le& \|\mathcal{T}\|^p \|f_1\|^p \\ =& \|\mathcal{T}\|^p \int_{|y-h|<\varepsilon/2} |f(x)|^p dx.\end{aligned}$$

And if $|x-h| < 1/4$, $\varepsilon/2 \le |y-h| < 5\,\varepsilon/4$, then $\varepsilon/4 < |x-y| < 3\,\varepsilon/2$, and then (4.1.3) gives trivially that

$$\int_{|x-h|<\varepsilon/4} |\mathcal{T}_\varepsilon f_2(x)|^p dx \le A\|f_2\|^p.$$

Finally, it is obvious that

$$\mathcal{T}_\varepsilon f_3(x) = 0$$

from $|x-h| < \varepsilon/4$ and $|y-h| < 5\,\varepsilon/4$. Combining those three estimates together yields that

$$\int_{|x-h|<\varepsilon/4} |\mathcal{T}_\varepsilon f(x)|^p dx \le C(A + \|\mathcal{T}\|)^p \int_{|y-h|<5\,\varepsilon/4} |f(y)|^p dy$$

holds uniformly for $h \in \mathbb{R}^n$. This means

$$\|\mathcal{T}_\varepsilon\|_p \leq C(A + \|\mathcal{T}\|)\|f\|_p.$$

■

Proof of Theorem 4.1.1 We wish to obtain

$$\|Tf\|_p \leq C(degP, n)\|f\|_p, \quad 1 < p < \infty. \tag{4.1.5}$$

We shall carry out the argument by a double induction on the degree in x and y of the polynomial P. If $P(x, y)$ depends only on x or only on y, it is obvious that these cases are trivial. Let k and l be two positive integers and let $P(x, y)$ have degree k in x and l in y. We assume that (4.1.5) holds for all polynomials which are sum of monomials of degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y.

We proceed to the proof of the inductive step. Rewrite

$$P(x, y) = \sum_{|\alpha|=k, |\beta|=l} a_{\alpha,\beta}\left(x^\alpha y^\beta - y^{\alpha+\beta}\right) + P_0(x, y)$$

and also write

$$P(x, y) = \sum_{|\alpha|=k} x^\alpha Q_\alpha(y) + R(x, y), \tag{4.1.6}$$

where $P_0(x, y)$ satisfies the inductive hypothesis. By dilation-invariance, we may assume that

$$\sum_{|\alpha|=u, |\beta|=v} |a_{\alpha,\beta}| = 1.$$

It is easy to see that when $|x| < 1$ and $|y| < 2$, we have

$$\left|e^{iP(x,y)} - e^{iP_0(x,y)}\right| \leq C|x - y|.$$

Decompose T as

$$\begin{aligned} Tf(x) = &\int_{|x-y|<1} e^{iP(x,y)} K(x - y)f(y)dy + \int_{|x-y|\geq 1} e^{iP(x,y)} K(x - y)f(y)dy \\ :=& T_0 f(x) + T_\infty f(x). \end{aligned}$$

We first consider T_0. We shall show that

$$\|T_0 f\|_p \leq C(degP, n)\|f\|_p. \tag{4.1.7}$$

Put $h \in \mathbb{R}^n$, and write

$$P(x,y) = \sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}(x-h)^\alpha (y-h)^\beta + R(x,y,h),$$

where the polynomial $R(x,y,h)$ satisfies the inductive hypothesis, and the coefficients of $R(x,y,h)$ depend on h. It follows that

$$\begin{aligned}
&|T_0 f(x)| \\
&\le \left| \int_{|x-y|<1} e^{i\left[\sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}(y-h)^{\alpha+\beta} + R(x,y,h)\right]} K(x-y) f(y) dy \right| \\
&\quad + \left| \int_{|x-y|<1} \left(e^{iP(x,y)} - e^{i\left[\sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}(y-h)^{\alpha+\beta} + R(x,y,h)\right]} \right) K(x-y) f(y) dy \right| \\
&:= |T_0^1 f(x)| + |T_0^2 f(x)|.
\end{aligned}$$

From the inductive hypothesis and Lemma 4.1.4 for truncated integral operator, we conclude that the operator T_0^1 is bounded on $L^p(\mathbb{R}^n)$, and the norm of T_0^1 is independent of P and h.

When $|x-h| < 1/4, |x-y| < 1$, we have

$$\left| e^{iP(x,y)} - e^{i\left[\sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}(y-h)^{\alpha+\beta} + R(x,y,h)\right]} \right| \le C|x-y|.$$

Thus, applying the properties of the kernel K, we conclude that

$$\begin{aligned}
|T_0^2 f(x)| &\le A \int_{|x-y|<1} \frac{|f(y)|}{|x-y|^{n-1}} dy \\
&\le A \int_{|y|<1} \frac{|f(x-y)\chi_{B(h,5/4)}(x-y)|}{|y|^{n-1}} dy.
\end{aligned}$$

By Minkowski's inequality, we obtain

$$\int_{|x-h|<1/4} |T_0^2 f(x)|^p dx \le A^p \int_{|y-h|<5/4} |f(y)|^p dy.$$

Therefore

$$\|T_0^2 f\|_p \le CA(degP, n)\|f\|_p.$$

Hence

$$\|T_0 f\|_p \le C(A+1)(degP, n)\|f\|_p,$$

where C is independent of the coefficients of $P(x,y)$.

Let us now turn our attention to T_∞. Denote

$$K_\infty = \sum_{j=0}^{\infty} \Psi_j,$$

where Ψ_0 is supported in $\{x : \frac{1}{2} \le |x| \le 1\}$ and is bounded, while $\Psi_j(x) = 2^{-nj}\Psi(2^{-j}x), j \ge 1$, with Ψ an appropriate function of class C^1 supported in $\{x : \frac{1}{2} \le |x| \le 1\}$. We set

$$T_j f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \Psi_j(x-y) f(y) dy. \tag{4.1.8}$$

Since estimating T_0 is trivial, we will consider T_j for $j \ge 1$, and claim that the L^2-operator norms satisfy the estimate

$$\|T_j\|_2 \le C 2^{-j\varepsilon}, \qquad \text{for some } \varepsilon > 0. \tag{4.1.9}$$

To do this we consider $T_j^* T_j$, where T_j^* is the adjoint operator of T_j. The kernel of this operator, $\tilde{L}_j(y,z)$, is given by

$$\int_{\mathbb{R}^n} e^{i(P(x,z)-P(x,y))} \Psi_j(x-z) \overline{\Psi}_j(x-y) dx.$$

Obviously, we will obtain the same norm if we replace

$$\tilde{L}_j(y,z) \text{ by } L_j(y,z) = 2^{nj} \tilde{L}_j(2^j y, 2^j z).$$

The result is then

$$L_j(y,z) = \int_{\mathbb{R}^n} e^{i(P(x,z)-P(x,y))} \Psi_0(x-z) \overline{\Psi}_0(x-y) dx. \tag{4.1.10}$$

A trivial estimate for L_j that follows from (4.1.10) shows

$$|L_j(y,z)| \le C \chi_{B_2}(y-z),$$

where χ_{B_2} is the characteristic function of a ball of radius 2. Now we make the changes of variables $x \to x+y$ in (4.1.10), and then get the $x-$integration in polar coordinates with $x = rx',\ r = |x|,\ |x'| = 1,\ dx = r^{n-1} dr d\sigma(x)$. We also write, after invoking (4.1.6),

$$P(2^j(x), 2^j z) = \sum_{|\alpha|=k} 2^{kj} x^\alpha Q_\alpha(2^j z) + R_j(x,z),$$

where R_j has $x-$degree strictly less than k. Similarly

$$P(2^j(x), 2^j y) = \sum_{|\alpha|=k} 2^{kj} x^\alpha Q_\alpha(2^j y) + R_j(x, y).$$

After these substitutions we get that

$$L_j(y, z) = \int_{S^{n-1}} \left(\int_0^1 e^{i(E+F)} \tilde{\Psi}(x, y, z) dr \right) d\sigma(x'), \tag{4.1.11}$$

where $\tilde{\Psi}(x, y, z) = \Psi_0(x - z + y)\overline{\Psi}_0(x) r^{n-1}$ with

$$E = (r2^j)^k \sum_{|\alpha|=k} (x')^\alpha (Q_\alpha(2^j z) - Q_\alpha(2^j y))$$

and

$$F = R_j(x, z) - R_j(x, y).$$

Note that F has degree strictly less than k, in r, while E is of degree k. Thus if we use the van der Corput Lemma (Lemma 4.1.3) for the integral inside (4.1.10), then we obtain

$$|L_j(y, z)| C \leq 2^{-j\delta} \left(\sum_{|\alpha|=k} \left| Q_\alpha(2^j z) - Q_\alpha(2^j y) \right| \right)^{-\delta k} \chi_{B_2}(y - z).$$

Since

$$\sum_{|\alpha|=k, |\beta|=l} |a_{\alpha,\beta}| = 1,$$

there is an $|\alpha_0| = k$ and $|\beta_0| = l$, such that $|a_{\alpha_0,\beta_0}| \geq c > 0$. However

$$Q_{\alpha_0}(2^j z) = 2^{jl} \sum_{|\beta|=l} \alpha_0 \beta z^\beta + S(z),$$

where $S(z)$ has $z-$degree strictly less than l, so

$$\int_{\mathbb{R}^n} |\tilde{L}_j(y, z)| dz \leq C 2^{-j\delta} \int_{|z-y|\leq 2} |Q_{\alpha_0}(2^j z) - Q_{\alpha_0}(2^j y)|^{-\delta k} dz,$$

and hence by Lemma 4.1.2

$$\sup_y \int |\tilde{L}_j(y, z)| dz \leq C 2^{-j\delta} 2^{-jl\delta/k} = C 2^{-j2\varepsilon}$$

if we choose δ sufficiently small to that $\delta/k < 1/l$. Similarly

$$\sup_z \int |\tilde{L}_j(y,z)|dy \leq C2^{-j2\varepsilon}.$$

This proves that $\|T^*T\| \leq C2^{-j2\varepsilon}$, and therefore (4.1.9) is valid. It is obvious, however, that the norms of T_j on L^1 or L^∞ are uniformly bounded. Thus by the interpolation with (4.1.9) we obtain that the L^p norms of T_j are exponential decreasing as $j \to \infty$, if $1 < p < \infty$, allowing us to sum $\sum T_j$ and concluding the proof of the theorem.

Theorem 4.1.1 shows that the operator T defined by (4.1.1) is of type (L^p, L^p), where $1 < p < \infty$. For the endpoint case ($p = 1$), we will formulate the following theorem.

Theorem 4.1.2 *The operator T defined by (4.1.1) is of weak type (L^1, L^1), with bound depending only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.*

The proof of Theorem 4.1.2 is more complicated and much longer. it will be convenient to divided it into several lemmas. For the purpose, we split the kernel K as $K = K_0 + K_\infty$, where $K_0(x) = K(x)$ if $|x| \leq 1$ and $K_\infty(x) = K(x)$ if $|x| > 1$. We consider the corresponding splitting $T = T_0 + T_\infty$:

$$T_0 f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,\,y)} K_0(x-y) f(y) dy,$$

$$T_\infty f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,\,y)} K_\infty(x-y) f(y) dy.$$

Next we shall prove for any $\lambda > 0$ and $f \in L^1$ that

$$\sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : |T_0 f(x)| > \lambda\}\right| \leq C\|f\|_1 \tag{4.1.12}$$

and

$$\sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : |T_\infty f(x)| > \lambda\}\right| \leq C\|f\|_1. \tag{4.1.13}$$

First, we shall prove that

$$\left|\{x \in B_1(0) : |T_0 f(x)| > \lambda\}\right| \leq C\lambda^{-1} \int_{|y|<2} |f(y)|dy, \tag{4.1.14}$$

where $B_r(x)$ denotes the closed ball with the center at x and radius $r > 0$. To prove the inequality (4.1.14), we first introduce a lemma.

Lemma 4.1.5 *Let $\mathcal{K}$ be an operator of the form:*

$$\mathcal{K}f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

where the kernel K satisfies $|K(x, y)| \le C|x - y|^{-n}$. For $f \in L^1$ and $\varepsilon > 0$, set

$$\mathcal{K}_\varepsilon f(x) = \text{p.v.} \int_{|x-y|<\varepsilon} K(x, y) f(y) dy.$$

If

$$\sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : |\mathcal{K}f(x)| > \lambda\}\right| \le C\|f\|_1,$$

then, there exists a constant C_n depending only on the dimension n such that

$$\sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : |\mathcal{K}_\varepsilon f(x)| > \lambda\}\right| \le C_n\|f\|_1.$$

Proof. Let us first show that

$$|\{x \in B(h, \varepsilon/4) : |\mathcal{K}_\varepsilon f(x)| > \lambda\}| \le C_n \lambda^{-1} \int_{|y-h|<5\varepsilon/4} |f(y)| dy \qquad (4.1.15)$$

uniformly for $h \in \mathbb{R}^n$. Integrating both sides of the inequality (4.1.15) with respect to h, we get the conclusion of Lemma 4.1.5. For any fixed $h \in \mathbb{R}^n$, we split f into three parts as

$$f(y) = \begin{cases} f_1(y), & |y - h| < \varepsilon/2, \\ f_2(y), & \varepsilon/2 \le |y - h| < 5\,\varepsilon/4, \\ f_3(y), & \text{otherwise}. \end{cases}$$

Note that if $|x - h| < \varepsilon/4$, then

$$\mathcal{K}_\varepsilon f_1(x) = \mathcal{K} f_1(x).$$

So by the assumption on $\mathcal{K}$,

$$\begin{aligned} |\{x \in B(h, \varepsilon/4) : |\mathcal{K}_\varepsilon f_1(x)| > \lambda\}| &= |\{x \in B(h, \varepsilon/4) : |\mathcal{K} f_1(x)| > \lambda\}| \\ &= |\{x : |\mathcal{K} f_1(x)| > \lambda\}| \\ &\le C_n \lambda^{-1} \|f_1\|_1 \\ &\le C_n \lambda^{-1} \int_{|y-h|<5\varepsilon/4} |f(y)| dy. \end{aligned}$$

And if $|x-h|<\varepsilon/4$, $\varepsilon/2\le|y-h|<5\,\varepsilon/4$, then $\varepsilon/4<|x-y|<3\,\varepsilon/2$, and therefore

$$|\mathcal{K}_\varepsilon f_2(x)|\le C_n\int_{|y-h|<5\varepsilon/4}|f(y)|dy.$$

Hence, by Chebyshev's inequality,

$$|\{x\in B(h,\varepsilon/4):|\mathcal{K}_\varepsilon f_2(x)|>\lambda\}|\le C_n\lambda^{-1}\int_{|y-h|<5\varepsilon/4}|f(y)|dy.$$

Finally, it follows that

$$\mathcal{K}_\varepsilon f_3(x)=0$$

from $|x-h|<\varepsilon/4$ and $|y-h|<5\,\varepsilon/4$. Combining those three estimates together yields (4.1.15). ■

We turn to the proof of (4.1.14). The method of the proof of (4.1.14) is similar to that in Theorem 4.1.1. Rewrite

$$P(x,y)=\sum_{|\alpha|=k,|\beta|=l}a_{\alpha,\beta}\left(x^\alpha y^\beta-y^{\alpha+\beta}\right)+P_0(x,y)$$

where $P_0(x,y)$ satisfies the inductive hypothesis. By dilation-invariance, we may assume that

$$\sum_{|\alpha|=u,|\beta|=v}|a_{\alpha,\beta}|=1.$$

If $|x|<1$ and $|y|<2$, then

$$\left|e^{iP(x,y)}-e^{iP_0(x,y)}\right|\le C|x-y|.$$

Hence,

$$\begin{aligned}|T_0f(x)|&\le\left|\int_{\mathbb{R}^n}e^{iP_0(x,\,y)}K_0(x-y)f(y)dy\right|+C\int_{|x-y|<1}|x-y|^{-n+1}|f(y)|dy\\&:=|Uf(x)|+|Vf(x)|.\end{aligned}$$

If $|x|\le1$, then we have that

$$Uf(x)=\int_{|x-y|<1}e^{iP_0(x,\,y)}K(x-y)(f\chi_{B(0,2)})(y)dy$$

and

$$Vf(x)=V(f\chi_{B(0,2)})(x).$$

By the inductive hypothesis on $P_0(x,\ y)$ and Lemma 4.1.5, it implies that U is bounded from L^1 to $L^{1,\infty}$. On the other hand, by Chebyshev's inequality we get that

$$|\{x \in B_1(0) : |Vf(x)| > \lambda\}| \le C_n\lambda^{-1} \int_{|y|<2} |f(y)|dy.$$

Combining these estimates of U and V implies (4.1.14). Similarly by the translation-invariance, for any $h \in \mathbb{R}^n$, we can easily get that

$$|\{x \in B_1(h) : |T_0 f(x)| > \lambda\}| \le C\lambda^{-1} \int_{|y-h|<2} |f(y)|dy. \tag{4.1.16}$$

Integrating both sides of the inequality in (4.1.16) with respect to h, we get (4.1.12).

Next we shall prove the inequality (4.1.13). Take a nonnegative $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\mathrm{supp}(\varphi) \subset \{1/2 \le |x| \le 2\}, \qquad \sum_{j=0}^{\infty} \varphi(2^{-j}x) = 1 \ \text{ if } \ |x| \ge 1.$$

Put $K_j(x,y) = \varphi(2^{-j}(x-y))K_\infty(x,y)$, where $K_\infty(x,y) = e^{iP(x,y)}K_\infty(x-y)$ and decompose

$$K_\infty(x,y) = \sum_{j=0}^{\infty} K_j(x,y).$$

Define

$$V_j f(x) = \int_{\mathbb{R}^n} K_j(x,y)f(y)dy \qquad \text{for} \quad j \ge 0$$

and

$$Vf(x) = \sum_{j=1}^{\infty} V_j f(x).$$

Then $T_\infty = V_0 + V$. Evidently V_0 is bounded on L^1, so it suffices to deal with V. By the Calderón-Zygmund decomposition at a positive λ, we have a collection $\{Q\}$ of non-overlapping closed dyadic cubes and functions g and b such that

$$f = g + b;$$

$$\lambda \le \frac{1}{|Q|} \int_Q |f| \le c\lambda;$$

$$|\cup Q| \le \frac{c\|f\|_1}{\lambda};$$
$$\|g\|_\infty \le c\lambda;$$
$$\|g\|_1 \le c\|f\|_1;$$
$$b = \sum_Q b_Q;$$
$$\operatorname{supp}(b_Q) \subset Q;$$
$$\int b_Q = 0$$

and

$$\|b_Q\|_1 \le c\lambda|Q|.$$

We set

$$B_i = \sum_{|Q|=2^{in}} b_Q \quad (i \ge 1), \qquad B_0 = \sum_{|Q|\le 1} b_Q.$$

Put $\mathcal{U} = \bigcup \tilde{Q}$, where $\tilde{Q}$ denotes the cube with the same center as Q and with sidelength 4 times that of Q. When $x \in \mathbb{R}^n \setminus \mathcal{U}$, we note that

$$\begin{aligned} Vb(x) &= \sum_{j=1}^{\infty} V_j\left(\sum_{i\ge 0} B_i\right)(x) \\ &= \sum_{j=1}^{\infty}\sum_{i\ge 0}\int_{\mathbb{R}^n} K_j(x,y)B_i(y)dy \\ &= \sum_{i\ge 0}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} K_j(x,y)B_i(y)dy. \end{aligned}$$

By the definitions of $K_j(x,y)$, B_i and $x \in \mathbb{R}^n \setminus \mathcal{U}$, if $i \ge j \ge 1$, then it easily implies that

$$\int_{\mathbb{R}^n} K_j(x,y)B_i(y)dy = 0.$$

Therefore, we have that

$$\begin{aligned} Vb(x) &= \sum_{i\ge 0}\sum_{j\ge i+1}\int_{\mathbb{R}^n} K_j(x,y)B_i(y)dy \\ &= \sum_{k\ge 1}\sum_{j\ge k}\int_{\mathbb{R}^n} K_j(x,y)B_{j-k}(y)dy \\ &= \sum_{k\ge 1}\sum_{j\ge k} V_j(B_{j-k})(x). \end{aligned}$$

To prove the inequality (4.1.13), we introduce Lemma 4.1.6.

Lemma 4.1.6 *There exists an $\varepsilon > 0$ such that, for any positive integer k,*

$$\left\| \sum_{j \geq k} V_j(B_{j-k}) \right\|_2^2 \leq c 2^{-\varepsilon k} \lambda \|f\|_1. \tag{4.1.17}$$

By (4.1.8), we have, for some $\varepsilon > 0$ and for all $j \geq 1$, that

$$\|V_j\|_2 \leq C 2^{-j\varepsilon}.$$

So we get that V is bounded on $L^2(\mathbb{R}^n)$.

Assume Lemma 4.1.6 is valid. We now prove that the inequality (4.1.13) holds.

$$\begin{aligned}
&\Big| \{x \in \mathbb{R}^n \setminus \mathcal{U} : |Vf(x)| > \lambda\} \Big| \\
&\leq \left| \left\{ x \in \mathbb{R}^n \setminus \mathcal{U} : |Vg(x)| > \frac{\lambda}{2} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n \setminus \mathcal{U} : |Vb(x)| > \frac{\lambda}{2} \right\} \right| \\
&\leq C\lambda^{-1} \|f\|_1 + C\lambda^{-2} \left\| \sum_{k \geq 1} \sum_{j \geq k} V_j(B_{j-k}) \right\|_2^2 \\
&\leq C\lambda^{-1} \|f\|_1 + C\lambda^{-2} \left(\sum_{k \geq 1} \lambda^{1/2} 2^{-\varepsilon k/2} \|f\|_1^{1/2} \right)^2 \\
&\leq C\lambda^{-1} \|f\|_1.
\end{aligned}$$

Thus we obtain that

$$\Big| \{x \in \mathbb{R}^n \setminus \mathcal{U} : |Vf(x)| > \lambda\} \Big| \leq C\lambda^{-1} \|f\|_1. \tag{4.1.18}$$

On the other hand, from the definition of $\mathcal{U}$ it follows that

$$|\mathcal{U}| \leq C\lambda^{-1} \|f\|_1. \tag{4.1.19}$$

Combining (4.1.18) with (4.1.19) yields the boundedness of V from L^1 to $L^{1,\infty}$. This is just our desired inequality (4.1.13).

It remains to show that the inequality (4.1.17) holds. For $k, m \geq 1$, put

$$\begin{aligned} & H_{km}(x,y) \\ &= \int_{\mathbb{R}^n} \overline{K}_k(z,x) K_m(z,y) dz \\ &= \int_{\mathbb{R}^n} e^{-iP(z,x)+iP(z,y)} \overline{K}_\infty(z-x) K_\infty(z-y) \varphi_k(z-x)\varphi_m(z-y) dz. \end{aligned} \tag{4.1.20}$$

Then

$$V_k^* V_m f(x) = \int_{\mathbb{R}^n} H_{km}(x,y) dy,$$

where V_k^* denotes the adjoint of V_k. We need to discuss the property of kernel H_{km} furthermore. To the end, we formulate the following Lemma 4.1.7.

Lemma 4.1.7 *Let $k \geq m \geq 1$. Then $H_{km}(x,y) = 0$, if $|x-y| > 2^{k+2}$; and*

(i) $|H_{km}(x,y)| \leq c2^{-kn}$,
(ii) $|H_{km}(x,y)| \leq c2^{-kn}2^{-m}|q(x)-q(y)|^{-1/M}$.

Proof. Suppose that $|x-y| > 2^{k+2}$. Since

$$|z-x| + |z-y| \geq |x-y| > 2^{k+2},$$

either of inequalities

$$|z-x| > 2^{k+1} \quad \text{and} \quad |z-y| > 2^{k+1}$$

must hold. That means $\varphi_k(z-x)\varphi_m(z-y) = 0$. Now we prove the estimate of (ii) only. Note that

$$\left(\frac{\partial}{\partial z}\right)^M \Big(P(z,x) - P(z,y)\Big) = M!\Big(q(x) - q(y)\Big).$$

Hence, it follows from Lemma 4.1.3 that

$$\left| \int_a^b e^{-iP(z,x)+iP(z,y)} dz \right| \leq c|q(x) - q(y)|^{-1/M}$$

for any a and b. Therefore by the integration by parts in variable z in the formula of (4.1.20) and by using the properties of the kernel K, we easily get the desired conclusion. ■

Let us give two definitions.

Definition 4.1.1 *For a real-valued polynomial* $P(x) = \sum_{|\alpha|\le N} a_\alpha x^\alpha$ *of degree* N*, define*

$$\|P\| = \max_{|\alpha|=N} |a_\alpha|.$$

Definition 4.1.2 *For a real-valued polynomial* P *and* $\beta > 0$*, let*

$$\mathcal{R}(P,\beta) = \{x \in \mathbb{R}^n : |P(x)| \le \beta\}.$$

Let $d(E,F)$ denote the distance between E and F and $d(x,F) = d(\{x\},F)$. Let us state the following Lemma which will be used to prove the Lemma 4.1.6.

Lemma 4.1.8 *Let* s, m *be integers and* $s \ge m$*. Suppose* $N \ge 1$*. Then, for any polynomial* P *of degree* N *satisfying* $\|P\| = 1$ *and for any* $\gamma > 0$*, there exists a positive constant* $C_{n,N,\gamma}$ *depending only on* n, N, γ *such that*

$$\left| \left\{x \in B_{2^k}(a) : d\left(x, \mathcal{R}(P, 2^{Nm})\right) \le \gamma 2^m\right\} \right| \le C_{n,N,\gamma} 2^{(n-1)s} 2^m$$

uniformly in $a \in \mathbb{R}^n$.

The proof of Lemma 4.1.8 needs to use geometrical properties of polynomials. For the details, one refers to [Sa] and we omit it here.

Let $\lambda > 0$ and let $\{\mathcal{B}_j\}_{j\ge 0}$ be family of measurable functions such that

$$\int_{Q_j} |\mathcal{B}_j| \le \lambda |Q_j|$$

for all cubes Q_j in $\mathbb{R}^n$ with sidelength $l(Q_j) = 2^j$.

Lemma 4.1.9 *Let the kernels* H_{ji} *be as in Lemma 4.1.7. Then we can find a constant* c *such that*

$$\sum_{i=k}^{j} \sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathcal{B}_j(y) H_{ji}(x,y)dy \right| \le c\lambda 2^{-k}$$

for all integers j *and* k *with* $0 < k \le j$.

Proof. For $m \in \mathbb{Z}$, let $\mathcal{D}_m$ be the family of all closed dyadic cubes Q with sidelength $l(Q) = 2^m$. Fix $x \in \mathbb{R}^n$. Let

$$\mathcal{F} = \{Q \in \mathcal{D}_{i-k} : Q \cap B_{2^{j+2}}(x) \neq \emptyset\} \quad (0 < k \leq i \leq j).$$

Then clearly $\sum_{Q\in\mathcal{F}} |Q| \leq c2^{jn}$. Decompose $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where

$$\mathcal{F}_0 = \left\{Q \in \mathcal{F} : Q \cap \mathcal{R}\left(q(\cdot) - q(x), 2^{L(i-k)}\right) \neq \emptyset\right\}$$

and $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$. Then by Lemma 4.1.8 we have

$$\sum_{Q\in\mathcal{F}_0} |Q| \leq c2^{j(n-1)}2^{i-k}.$$

It implies by (i) in Lemma 4.1.7 and above estimates that

$$\begin{aligned}\sum_{Q\in\mathcal{F}_0} \int_Q |\mathcal{B}_{i-k}(y)H_{ji}(x,y)|dy &\leq c2^{-jn} \sum_{Q\in\mathcal{F}_0} \int_Q |\mathcal{B}_{i-k}(y)|dy \\ &\leq c2^{-jn}\lambda \sum_{Q\in\mathcal{F}_0} |Q| \leq c\lambda 2^{i-j-k}.\end{aligned} \tag{4.1.21}$$

From (ii) in Lemma 4.1.7 and above estimates, it follows that

$$\begin{aligned}&\sum_{Q\in\mathcal{F}_1} \int_Q |\mathcal{B}_{i-k}(y)H_{ji}(x,y)|dy \\ &\leq c2^{-jn}2^{-i}2^{-L(i-k)/M} \sum_{Q\in\mathcal{F}_1} \int_Q |\mathcal{B}_{i-k}(y)|dy \\ &\leq c2^{-jn}2^{-i}2^{-L(i-k)/M}\lambda \sum_{Q\in\mathcal{F}} |Q| \leq c\lambda 2^{-i}2^{-L(i-k)/M}.\end{aligned} \tag{4.1.22}$$

From (4.1.21) and (4.1.22) it follows that

$$\begin{aligned}&\int_{\mathbb{R}^n} |\mathcal{B}_{i-k}(y)H_{ji}(x,y)|dy \\ &= \sum_{Q\in\mathcal{F}} \int_Q |\mathcal{B}_{i-k}(y)H_{ji}(x,y)|dy \\ &= \sum_{Q\in\mathcal{F}_0} \int_Q |\mathcal{B}_{i-k}(y)H_{ji}(x,y)|dy + \sum_{Q\in\mathcal{F}_1} \int_Q |\mathcal{B}_{i-k}(y)H_{ji}(x,y)|dy \\ &\leq c\lambda\left(2^{i-j-k} + 2^{-i}2^{-L(i-k)/M}\right).\end{aligned}$$

Thus we see that

$$\sum_{i=k}^{j} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{B}_j(y)H_{ji}(x,y)|\, dy \le \sum_{i=k}^{j} c\lambda \left(2^{i-j-k} + 2^{-i}2^{-L(i-k)/M}\right) \le c\lambda 2^{-k}.$$

This completes the proof of Lemma 4.1.9. ■

Lemma 4.1.10 *Let $\{\mathcal{B}_j\}_{j\ge 0}$ be as in Lemma 4.1.9. Suppose $\sum_{j\ge 0} \|\mathcal{B}_j\|_1 < \infty$. Then, for any positive integer k, we have*

$$\left\| \sum_{j\ge k} V_j(\mathcal{B}_{j-k}) \right\|_2^2 \le c\lambda 2^{-k} \sum_{j\ge 0} \|\mathcal{B}_j\|_1.$$

Proof. Let $\langle \cdot, \cdot \rangle$ denote the inner product in L^2. Using Lemma 4.1.9, we note that

$$\begin{aligned} \left\| \sum_{j\ge k} V_j(\mathcal{B}_{j-k}) \right\|_2^2 &\le 2 \sum_{j\ge s} \sum_{i=k}^{j} |\langle V_j(\mathcal{B}_{j-k}), V_i(\mathcal{B}_{i-k})\rangle| \\ &= 2 \sum_{j\ge s} \sum_{i=k}^{j} \left|\left\langle \mathcal{B}_{j-k}, V_j^* V_i(\mathcal{B}_{i-k})\right\rangle\right| \\ &\le 2 \sum_{j\ge k} \sum_{i=k}^{j} \|\mathcal{B}_{j-k}\|_1 \left\| V_j^* V_i(\mathcal{B}_{i-k})\right\|_{L^\infty} \\ &\le c\lambda 2^{-k} \sum_{j\ge 0} \|\mathcal{B}_j\|_1. \end{aligned}$$

This completes the proof of Lemma 4.1.10. ■

Set $\mathcal{B}_j = B_j$, then we have that

$$\sum_{j\ge 0} \|\mathcal{B}_j\|_1 = \sum_{j\ge 0} \|B_j\|_1 \le \sum_{j\ge 1} c\lambda \sum_{|Q_j|=2^{jn}} |Q_j| + c\lambda \sum_{|Q|\le 1} |Q| \le c\|f\|_1.$$

Thus we obtain that

$$\left\| \sum_{j\ge k} V_j(\mathcal{B}_{j-k}) \right\|_2^2 \le c\lambda 2^{-k} \|f\|_1,$$

which is just the inequality (4.1.17). This completes the proof of Theorem 4.1.2.

4.2 Oscillatory singular integrals with rough kernels

In this section, we will investigate a class of oscillatory singular integral operators with rough kernels. Suppose that $K(x) = \dfrac{\Omega(x')}{|x|^n}$ satisfies conditions

(1) Ω is homogeneous of degree 0;

(2) $\displaystyle\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0$; and

(3) $\Omega \in L^q(\mathbb{S}^{n-1})$, for some q with $1 < q \leq \infty$.

Let us introduce a truncated operator

$$Sf(x) = \int_{|x-y|<1} K(x-y)f(y)dy. \tag{4.2.1}$$

Theorem 4.2.1 *Suppose that $K(x)$ satisfies (1) and (3). The operator T defined by (4.1.1) is bounded on $L^p(\mathbb{R}^n)$ if and only if the truncated operator S is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with the bound depending only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.*

Proof. Assume that a polynomial $P(x,y)$ has degree k in x and l in y and $\displaystyle\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}| = 1$. Let us first consider the following operator T_∞ defined by

$$T_\infty f(x) = \int_{|x-y|\geq 1} e^{iP(x,y)}K(x-y)f(y)dy.$$

Write

$$T_\infty f(x) = \sum_{j=1}^{\infty} \int_{2^{j-1}\leq|x-y|<2^j} e^{iP(x,y)}K(x-y)f(y)dy := \sum_{j=1}^{\infty} T_j f(x),$$

where

$$\begin{aligned} T_j f(x) &= \int_{2^{j-1}\leq|y|<2^j} e^{iP(x,x-y)}K(y)f(x-y)dy \\ &= \int_{\mathbb{S}^{n-1}} \Omega(y') \int_{2^{j-1}\leq r<2^j} e^{iP(x,x-ry')}\frac{f(x-ry')}{r}drd\sigma(y'). \end{aligned}$$

For a fixed $y' \in \mathbb{S}^{n-1}$, let Y be the hyperplane through the origin orthogonal to y'. We have, for $x \in \mathbb{R}^n$, $x = z + sy'$ with $s \in \mathbb{R}$ and $z \in Y$,

that

$$\begin{aligned}
&\int_{2^{j-1}\le r<2^j} e^{iP(x,x-ry')}\frac{f(x-ry')}{r}dr\\
&=\int_{2^{j-1}\le r<2^j} e^{iP(z+sy',z+(s-r)y')}\frac{f(z+(s-r)y')}{r}dr\\
&=\int_{2^{j-1}\le s-t<2^j} e^{iP(z+sy',z+ty')}\frac{f(z+ty')}{s-t}dt\\
&=N_j[f(z+\cdot y')](s).
\end{aligned}$$

It is easy to see that N_j is a linear operator defined on $L^2(\mathbb{R})$. Denote N_j^* to be its adjoint operator. Next we will consider the operator $N_j^*N_j$ with the kernel

$$\begin{aligned}
M_j(u,v) &= \int_{2^{j-1}\le r-v,r-u<2^j} \frac{e^{i[P(z+ry',z+vy')-P(z+ry',z+uy')]}}{(r-u)(r-v)}dr\\
&= \int_{\frac{1}{2}\le r<1,2^{j-1}\le r+v-u<2^j} \frac{e^{i[P(z+2^jry'+vy',z+vy')-P(z+2^jry'+vy',z+uy')]}}{r(2^jr+v-u)}dr.
\end{aligned}$$

It is easy to check that

$$|M_j(u,v)| \le \frac{1}{2^j}\chi_{[0,2^{j-1})}(|v-u|). \tag{4.2.2}$$

Now rewrite $P(x,y)$ as

$$P(x,y)=\sum_{|\alpha|=k} x^\alpha Q_\alpha(y)+R(x,y).$$

Thus we have that

$$M_j(u,v)=\int_{\frac{1}{2}\le r<1,2^{j-1}\le r+v-u<2^j} e^{i(E+F)}\psi(r)dr,$$

where

$$E=(2^jr)^k\sum_{|\alpha|=k} y'^\alpha\Big[Q_\alpha(z+vy')-Q_\alpha(z+uy')\Big],$$

F with $r-$degree less than k, and

$$\psi(r)=\frac{1}{r(2^jr+v-u)}.$$

It follows from Lemma 4.1.3 that

$$\left|\int_{\frac{1}{2}\leq r<1} e^{i(E+F)}dr\right| \leq C\left(2^{jk}\left|\sum_{|\alpha|=k} y'^{\alpha}\Big[Q_\alpha(z+vy')-Q_\alpha(z+uy')\Big]\right|\right)^{-\frac{1}{k}}.$$

It follows from integration by parts that

$$|M_j(u,v)| \leq C2^{-j}\left(2^{jk}\left|\sum_{|\alpha|=k} y'^{\alpha}\Big[Q_\alpha(z+vy')-Q_\alpha(z+uy')\Big]\right|\right)^{-\frac{1}{k}}. \tag{4.2.3}$$

Combining the inequality (4.2.2) with (4.2.3) yields the following estimate

$$\begin{aligned}&|M_j(u,v)|\\ &\leq C2^{-j}\left(2^{jk}\left|\sum_{|\alpha|=k} y'^{\alpha}\Big[Q_\alpha(z+vy')-Q_\alpha(z+uy')\Big]\right|\right)^{-\frac{\delta}{k}} \chi_{[0,2^j-1)}(|v-u|)\end{aligned}$$

holds uniformly in $\delta\in(0,1]$. Thus

$$\begin{aligned}&\int |M_j(u,v)|dv\\ &\leq C2^{-j}2^{-j\delta}\int_{|u-v|<2^j}\left(2^{jk}\left|\sum_{|\alpha|=k} y'^{\alpha}\Big[Q_\alpha(z+vy')-Q_\alpha(z+uy')\Big]\right|\right)^{-\frac{\delta}{k}} dv\\ &\leq C2^{-j\delta}\int_{|v|<1}\left(2^{jk}\left|\sum_{|\alpha|=k} y'^{\alpha}\Big[Q_\alpha(z+(v+\frac{u}{2^j}y')-Q_\alpha(z+uy')\Big]\right|\right)^{-\frac{\delta}{k}} dv.\end{aligned}$$

Now we take $\delta\in(0,1]$ such that $\delta/k<1/l$, then from Lemma 4.1.1 it follows that

$$\int |M_j(u,v)|dv \leq C2^{-j\delta}\left|\sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}y'^{\alpha+\beta}\right|^{-\frac{\delta}{k}}.$$

Thus, we have that

$$\|N_j^*N_j|\|_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})} \leq C2^{-j\delta}\left|\sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}y'^{\alpha+\beta}\right|^{-\frac{\delta}{k}}$$

and

$$\|N_j^* N_j|\|_{L^1(\mathbb{R})\to L^1(\mathbb{R})} \le C2^{-j\delta}\left|\sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}{y'}^{\alpha+\beta}\right|^{-\frac{\delta}{k}}.$$

By the Riesz-Thörin interpolation theorem, we obtain that

$$\|N_j^* N_j|\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le C2^{-j\delta}\left|\sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}{y'}^{\alpha+\beta}\right|^{-\frac{\delta}{k}}.$$

Since

$$|N_j g(s)| \le \int_{2^{j-1}\le s-t<2^j} \frac{|g(t)|}{s-t}dt \le 4HL(g)(s),$$

we have that

$$\|N_j|\|_{L^{p_0}(\mathbb{R})\to L^{p_0}(\mathbb{R})} \le C(p_0), \quad \text{with} \quad 1<p_0<\infty.$$

By the Riesz-Thörin interpolation theorem again, we obtain that

$$\|N_j|\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \le C2^{-\theta j\delta/2}\left|\sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}{y'}^{\alpha+\beta}\right|^{-\frac{\theta\delta}{2k}}, \tag{4.2.4}$$

where $0<\theta<1$. From Minkowski's inequality and (4.2.4), it follows that

$$\begin{aligned}
&\|T_j f\|_p \\
&\le \int_{\mathbb{S}^{n-1}} |\Omega(y')| \left(\int_Y \int_{\mathbb{R}} |N_j[f(z+\cdot y')](s)|^p ds dz\right)^{1/p} d\sigma(y') \\
&\le C2^{-\theta j\delta/2}\|f\|_p \int_{\mathbb{S}^{n-1}} |\Omega(y')| \left|\sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}{y'}^{\alpha+\beta}\right|^{-\frac{\theta\delta}{2k}} d\sigma(y') \\
&\le C2^{-\theta j\delta/2}\|f\|_p\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left(\int_{\mathbb{S}^{n-1}} \left|\sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}{y'}^{\alpha+\beta}\right|^{-\frac{\theta\delta q'}{2k}} d\sigma(y')\right)^{1/q'}.
\end{aligned}$$

We take $\delta\in(0,1]$ such that $\delta<\min\{k/l, 2k/(k+l)q'\}$. By Lemma 4.1.2, we obtain that

$$\|T_j f\|_p \le C2^{-\theta j\delta/2}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_p.$$

Thus

$$\|T_\infty f\|_p \le C\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_p.$$

If $\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}| \neq 1$, we denote

$$A = \left(\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}| \right)^{1/(k+l)}.$$

We can write $P(x,y)$ as

$$P(x,y) = \sum_{|\alpha|=k,|\beta|=l} \frac{a_{\alpha\beta}}{A^{k+l}}(Ax)^\alpha (Ay)^\beta + R_0\left(\frac{Ax}{A}, \frac{Ay}{A}\right) := Q(Ax, Ay).$$

Thus

$$\begin{aligned} T_\infty f(x) &= \int_{|x-y|\ge 1} e^{iQ(Ax,Ay)} K(x-y) f(y) dy \\ &= \int_{|x-\frac{y}{A}|\ge 1} e^{iQ(Ax,y)} K(Ax-y) f\left(\frac{y}{A}\right) dy. \end{aligned}$$

So we have that

$$\begin{aligned} \|T_\infty f\|_p &= \left(\int_{\mathbb{R}^n} \left| \int_{|x-\frac{y}{A}|\ge 1} e^{iQ(Ax,y)} K(Ax-y) f\left(\frac{y}{A}\right) dy \right|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^n} A^{-n} \left| \int_{|x-y|\ge A} e^{iQ(x,y)} K\left(x-y\right) f\left(\frac{y}{A}\right) dy \right|^p dx \right)^{\frac{1}{p}} \\ &\le CA^{-n/p} \left\| f\left(\frac{(\cdot)}{A}\right) \right\|_p \\ &\le C\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_p. \end{aligned}$$

Rewrite

$$Tf(x) = \text{p.v.} \int_{|x-y|<1} e^{iP(x,\, y)} K(x-y) f(y) dy + T_\infty f(x) := T_0 f(x) + T_\infty f(x).$$

Note that we have proved that T_∞ is a (L^p, L^p) type operator. If T is a (L^p, L^p) type operator, then T_0 is also of (L^p, L^p) type.

Let us now turn to prove that S is a (L^p, L^p) type operator. To do this, we first introduce a notion. For a polynomial P, we say that P has property $\mathscr{P}$, if it satisfies

$$P(x,y) = P(x-h, y-h) + R_0(x,h) + R_1(y,h),$$

where R_0 , R_1 are real polynomials.

We take $h \in \mathbb{R}^n$. For $|x-h| < 1$, we have

$$T_0 f(x) = T_0[f(\cdot)\chi_{B_2(h)}(\cdot)](x).$$

Thus

$$\left(\int_{|x-h|<1} |T_0 f(x)|^p dx\right)^{1/p} \le C\left(\int_{|y-h|<2} |f(y)|^p dx\right)^{1/p}, \tag{4.2.5}$$

where C is independent of h.

We choose polynomial P with the property $\mathscr{P}$. It follows that

$$\begin{aligned} Sf(x) &= \int_{|x-y|<1} K(x-y)f(y)\chi_{B_2(h)}(y)dy \\ &= e^{-iR_0(x,h)}\int_{|x-y|<1} e^{iP(x,y)}K(x-y) \\ &\quad \times e^{-iP(x-h,y-h)}e^{-iR_1(y,h)}f(y)\chi_{B_2(h)}(y)dy. \end{aligned}$$

Note that the Taylor expression of $e^{iP(x-h,y-h)}$ is

$$\begin{aligned} e^{-iP(x-h,y-h)} &= \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\left(\sum_{\alpha,\beta} a_{\alpha\beta}(x-h)^\alpha(y-h)^\beta\right)^m \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{\mu,\nu,} a_{m,\mu,\nu}(x-h)^\mu(y-h)^\nu. \end{aligned}$$

Thus, if we set $a = (1, 1, \cdots, 1)$, $b = (2, 2, \cdots, 2)$, then we conclude that

$$\begin{aligned}
&\left(\int_{|x-h|<1} |Sf(x)|^p dx\right)^{1/p} \\
&\le \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} |a_{m,\mu,\nu}| \Big(\int_{|x-h|<1} |(x-h)^\mu|^p \\
&\quad \times \left| T_0 \left(e^{-iR_1(\cdot,h)} f(\cdot)\chi_{B(h,2)}(\cdot)(\cdot - h)^\nu \right)(x) \right|^p dx \Big)^{1/p} \\
&\le \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} |a_{m,\mu,\nu}| a^\mu \left(\int_{|y-h|<2} |f(y)|^p |(y-h)^\nu|^p dy \right)^{1/p} \\
&\le \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu} |a_{m,\mu,\nu}| a^\mu b^\nu \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p} \\
&\le C \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta \right)^m \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p} \\
&\le C \exp\left(\sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta \right) \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p}.
\end{aligned}$$

Thus

$$\|Sf\|_p \le C\|f\|_p.$$

On the other hand, we assume that the truncated operator S is bounded on $L^p(\mathbb{R}^n)$. Let us now turn to prove that operator T is also bounded on $L^p(\mathbb{R}^n)$. As proved above, T_∞ is bounded on $L^p(\mathbb{R}^n)$. In fact, it suffices to check that T_0 is also bounded on $L^p(\mathbb{R}^n)$. We shall carry out the argument by a double induction on the degree in x and y of the polynomial P. If $P(x, y)$ depends only on x or only on y, it is clear that there cases are trivial. Let k and l be two positive integers and let $P(x, y)$ have degree k in x and l in y. We assume that T_0 is bounded on $L^p(\mathbb{R}^n)$ for all polynomial which are sums of monomials of degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y.

We proceed now to the proof of the inductive step. Rewrite

$$P(x, y) = \sum_{|\alpha|=k, |\beta|=l} a_{\alpha,\beta} \left(x^\alpha y^\beta - y^{\alpha+\beta} \right) + P_0(x, y)$$

and also write

$$P(x,y) = \sum_{|\alpha|=k} x^\alpha Q_\alpha(y) + P_0(x,y),$$

where $P_0(x,y)$ satisfies the inductive hypothesis. By dilation-invariance, we may assume that

$$\sum_{|\alpha|=u, |\beta|=v} |a_{\alpha,\beta}| = 1.$$

Take $h \in \mathbb{R}^n$, and write

$$P(x,y) = \sum_{|\alpha|=k, |\beta|=l} a_{\alpha,\beta}(x-h)^\alpha (y-h)^\beta + R(x,y,h),$$

where the polynomial $R(x,y,h)$ satisfies the inductive hypothesis, and the coefficients of $R(x,y,h)$ depend on h. It follows that

$$\begin{aligned}
&|T_0 f(x)| \\
&\leq \left| \int_{|x-y|<1} e^{i\left[\sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}(y-h)^{\alpha+\beta} + R(x,y,h)\right]} K(x-y) f(y) dy \right| \\
&\ + \left| \int_{|x-y|<1} \left(e^{iP(x,y)} - e^{i\left[\sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}(y-h)^{\alpha+\beta} + R(x,y,h)\right]} \right) K(x-y) f(y) dy \right| \\
&:= |T_0^1 f(x)| + |T_0^2 f(x)|.
\end{aligned}$$

By the inductive hypothesis, we conclude that the operator T_0^1 is bounded on $L^p(\mathbb{R}^n)$, and the norm of T_0^1 is independent of $P(x,y)$ and h. When $|x-h| < 1/4, |x-y| < 1$, we have

$$\left| e^{iP(x,y)} - e^{i\left[\sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}(y-h)^{\alpha+\beta} + R(x,y,h)\right]} \right| \leq C|x-y|.$$

Thus, we conclude that

$$\begin{aligned}
|T_0^2 f(x)| &\leq C \int_{|x-y|<1} \frac{|\Omega((x-y)')|}{|x-y|^{n-1}} |f(y)| dy \\
&\leq \int_{|y|<1} \frac{|\Omega(y')|}{|y|^{n-1}} |f(x-y) \chi_{B(h,5/4)}(x-y)| dy.
\end{aligned}$$

By Minkowski's inequality, we obtain

$$\int_{|x-h|<1/4} |T_0^2 f(x)|^p dx \leq C \|\Omega\|^p_{L^q(\mathbb{S}^{n-1})} \int_{|x-h|<5/4} |f(x)|^p dx.$$

Therefore

$$\|T_0^2 f\|_p \le CA(\deg P, n)\|f\|_p.$$

Hence

$$\|T_0 f\|_p \le C\|\Omega\|^p_{L^q(\mathbb{S}^{n-1})}\|f\|_p,$$

where C is independent of the coefficients of $P(x,y)$. ■

Theorem 4.2.1 tells us that the boundedness of an oscillatory singular integral operator is attributed to the boundedness of the corresponding truncated integral operator. The following theorem is an important application of Theorem 4.2.1.

Theorem 4.2.2 *Suppose that $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ and K satisfies the condition (1),(2) and (3). Then the operator T defined by (4.1) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with the bound depending only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.*

Proof. When K satisfies the condition (1), (2) and (3), By Theorem 2.3.7, we know that the operator

$$\mathscr{K} f(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

is a (L^p, L^p) type operator with $1 < p < \infty$. It suffices to prove that the truncated operator S is a (L^p, L^p) type operator with $1 < p < \infty$. For any fixed $h \in \mathbb{R}^n$, we split f into three parts as

$$f(y) = \begin{cases} f_1(y), & |y-h| < 1/2, \\ f_2(y), & 1/2 \le |y-h| < 5/4, \\ f_3(y), & \text{otherwise.} \end{cases}$$

It is easy to see that if $|x-h| < 1/4$, then we have

$$Sf_1(x) = \mathscr{K} f_1(x).$$

So we have that

$$\begin{aligned} \int_{|x-h|<1/4} |Sf_1(x)|^p dx &= \int_{|x-h|<1/4} \left| \int_{\mathbb{R}^n} K(x-y) f_1(y)dy \right|^p dx \\ &\le \int_{\mathbb{R}^n} |\mathscr{K} f_1(x)|^p dx \\ &\le \|\mathscr{K}\|^p \int_{|y-h|<1/2} |f(x)|^p dx. \end{aligned}$$

And if $|x-h| < 1/4$, $1/2 \leq |y-h| < 5/4$, then $1/4 < |x-y| < 3/2$. It follows from Minkowski's inequality that

$$\begin{aligned}
&\left(\int_{|x-h|<1/4} |Sf_2(x)|^p dx\right)^{1/p} \\
&\leq \int_{1/4<|y|\leq 1} \frac{|\Omega(y')|}{|y|^n} \left(\int_{|x-h|<1/4} |f_2(x-y)|^p dx\right)^{1/p} dy \\
&\leq C\|\Omega\|^p_{L^q(\mathbb{S}^{n-1})} \left(\int_{|x-h|<5/4} |f(x)|^p dx\right)^{1/p} dy.
\end{aligned}$$

Finally, it is clear that

$$Sf_3(x) = 0.$$

The above estimates imply that

$$\|Sf\|_p \leq C(\|\mathscr{K}\| + 1)\|f\|_p.$$

It immediately follows from Theorem 4.2.1 that operator T defined by (4.1) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with the bound depending only on the total degree of $P(x, y)$, but not on the coefficients of $P(x, y)$. ■

Now we weaken the condition on the kernel Ω. In place of $L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$, assume that Ω satisfies

(4) $\quad \Omega \in L \log L^+(\mathbb{S}^{n-1})$.

Obviously, the following theorem will weaken the condition on the kernel Ω in Theorem 4.2.1.

Theorem 4.2.3 *Suppose that K satisfies (1) and (4). The operator T defined by (4.1.1) is bounded on $L^p(\mathbb{R}^n)$ if and only if the truncated operator S is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with the bound depending only on the total degree of $P(x, y)$, but not on the coefficients of $P(x, y)$.*

Proof. The idea and method of proving Theorem 4.2.3 are much similar to that of Theorem 4.2.1. We only give the different part.

For $\Omega \in L \log L^+(\mathbb{S}^{n-1})$, we decompose the kernel Ω as follows:

$$\Omega_m(x') = \Omega(x')\chi_{E_m}(x'),$$

where

$$E_0 = \{x' \in \mathbb{S}^{n-1} : |\Omega(x')| < 1\}$$

and

$$E_m = \{x' \in \mathbb{S}^{n-1} : 2^{m-1} \le |\Omega(x')| < 2^m\}, \quad m \in \mathbb{N}.$$

We split T_∞ as follows:

$$T_\infty f(x) = \sum_{j=1}^{\infty}\sum_{m=0}^{\infty} \int_{2^{j-1}\le|x-y|<2^j} e^{iP(x,y)} \frac{\Omega_m(x-y)}{|x-y|^n} f(y)dy := \sum_{j=1}^{\infty} T_{j,m}f(x).$$

By a method similar to that of proving Theorem 4.2.1, we can prove that there exists a $\delta > 0$ such that

$$\|T_{j,m}f\|_p \le C2^{-j\delta}\|\Omega_m\|_{L^\infty(\mathbb{S}^{n-1})}\|f\|_p, \tag{4.2.6}$$

where C is independent of j and m. On the other hand, we obtain that

$$|T_{j,m}f(x)| \le \int_{2^{j-1}\le|y|<2^j} \frac{|\Omega_m(y)|}{|y|^n}\chi_{E_m}(y)|f(x-y)|dy.$$

It is clear that

$$\frac{|\Omega_m(y)|}{|y|^n}\chi_{E_m}(y) \in L^1(\mathbb{R}^n).$$

So we have that

$$\|T_{j,m}f\|_p \le C\|\Omega_m\|_{L^1(\mathbb{S}^{n-1})}\|f\|_p. \tag{4.2.7}$$

Now we choose a positive integer $M > \delta^{-1}$. Thus

$$\begin{aligned}\|T_\infty f\|_p &= \sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\|T_{j,m}f\|_p \\ &= \sum_{j=1}^{\infty}\|T_{j,0}f\|_p + \sum_{m=1}^{\infty}\sum_{j=1}^{mM}\|T_{j,m}f\|_p + \sum_{m=1}^{\infty}\sum_{j>mM}\|T_{j,m}f\|_p.\end{aligned}$$

It follows from (4.2.6) that

$$\sum_{j=1}^{\infty}\|T_{j,0}f\|_p \le C2^{-j\theta\delta}\|f\|_p$$

and

$$\sum_{m=1}^{\infty}\sum_{j>mM}\|T_{j,m}f\|_p \le C\sum_{m=1}^{\infty}\sum_{j>mM} 2^{-j\theta\delta}2^{m\theta}\|T_{j,m}f\|_p \le C\|f\|_p.$$

By (4.2.7), we obtain that

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{j=1}^{mM}\|T_{j,m}f\|_p &\le C\sum_{m=1}^{\infty}\sum_{j=1}^{mM}\|\Omega_m\|_{L^1(\mathbb{S}^{n-1})}\|f\|_p\\ &\le CM\|f\|_p\sum_{m=1}^{\infty}m2^m|E_m|\\ &\le CM\|\Omega\|_{L^1\log^+L(\mathbb{S}^{n-1})}\|f\|_p\\ &\le C\|f\|_p.\end{aligned}$$

Therefore we have that

$$\|T_\infty f\|_p \le C\|f\|_p.$$

The remainder of the proof is the same as that of Theorem 4.2.1, and we omit it here. ∎

The following Theorem is a direct consequence of Theorem 4.2.3.

Theorem 4.2.4 *Suppose that $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n\times\mathbb{R}^n$ and K satisfies the condition (1),(2) and (4). The operator T defined by (4.1) is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$, with the bound depending only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.*

4.3 Oscillatory singular integrals with standard kernels

Let $K(x,y)$ be a Calderón-Zygmund standard kernel (also called distribution kernel). That means, $K(x,y)$ satisfies

$$(a)\quad |K(x,y)|\le\frac{C}{|x-y|^n},\quad x\ne y,$$

$$(b)\quad |\nabla_x K(x,y)|+|\nabla_y K(x,y)|\le\frac{C}{|x-y|^{n+1}},\ \ x\ne y.$$

Clearly the kernel K does not have any homogeneous property.

The oscillatory singular integral operator in this section is defined by

$$Tf(x)=\text{p.v.}\int_{\mathbb{R}^n}e^{iP(x,y)}K(x,y)f(y)dy, \tag{4.3.1}$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n\times\mathbb{R}^n$.

Theorem 4.3.1 *Suppose that K satisfies (a) and (b). If the singular integral operator $\mathcal{K}$ defined by*

$$\mathcal{K}f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

is bounded on $L^2(\mathbb{R}^n)$, then T defined by (4.3.1) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with the bound depending only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.

We only state the above theorem without proof and will give the proof for its weighted version. In weighted case, we have the following theorem.

Theorem 4.3.2 *Under the same conditions on K as that of Theorem 4.3.1. If the operator $\mathcal{K}$ is bounded on $L^2(\mathbb{R}^n)$, then for any $\omega \in A_p$, $1 < p < \infty$, we have*

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega}, \tag{4.3.2}$$

where A_p is the Muckenhoupt class, and C depends only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.

Proof. Using same double induction on the degree in x and y of the polynomial P as that of Theorem 4.1.1. We consider two cases.

Case 1. Assume $\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}| = 1$. Decompose

$$\begin{aligned} Tf(x) = & \int_{|x-y|<1} e^{iP(x,y)}K(x,y)f(y)dy \\ & + \sum_{j=1}^{\infty} \int_{2^{j-1}\le|x-y|<2^j} e^{iP(x,y)}K(x,y)f(y)dy \\ := & T_0f(x) + \sum_{j=1}^{\infty} T_jf(x). \end{aligned}$$

For $h \in \mathbb{R}^n$, $P(x,y)$ can be written as

$$P(x,y) = \sum_{|\alpha|=k,|\beta|=l} a_{\alpha\beta}(x-h)^\alpha(y-h)^\beta + R(x,y,h),$$

where $R(x,y,h)$ is a polynomial which satisfies the inductive hypothesis, whose coefficients depend on h. Write

$$\begin{aligned}T_0f(x) =& \int_{|x-y|<1} \exp\left\{i\left(R(x,y,h) + \sum a_{\alpha\beta}(y-h)^{\alpha+\beta}\right)\right\} K(x,y)f(y)dy \\ &+ \int_{|x-y|<1} e^{iP(x,y)} - e^{i\left(R(x,y,h)+\sum a_{\alpha\beta}(y-h)^{\alpha+\beta}\right)} K(x,y)f(y)dy \\ :=& T_{01}f(x) + T_{02}f(x).\end{aligned}$$

Now put f into three parts:

$$\begin{aligned}f(y) =&f(y)\chi_{\{|y-h|<1/2\}}(y) + f(y)\chi_{\{1/2\le|y-h|<5/4\}}(y) + f(y)\chi_{\{|y-h|\ge 5/4\}}(y) \\ :=&f_1(y) + f_2(y) + f_3(y).\end{aligned}$$

When $|x-h| < 1/4$, it is easy to see that

$$T_{01}f_1(x) = \int_{\mathbb{R}^n} \exp\left\{i(R(x,y,h) + \Sigma a_{\alpha\beta}(y-h)^{\alpha+\beta})\right\} K(x,y)f_1(y)dy.$$

From the inductive hypothesis, it follows that

$$\int_{|x-h|<1/4} |T_{01}f_1(x)|^p\omega(x)dx \le C\int_{|y-h|<1/2} |f(y)|^p\omega(y)dy, \qquad (4.3.3)$$

where C does not depend on h and the coefficients of $P(x,y)$.

Note that when $|x-h| < 1/4, 1/2 < |y-h| < 5/4$, we have $|x-y| > 1/4$. Thus

$$|T_{01}f_2(x)| \le C\int_{1/4<|x-y|<1} \frac{|f_2(y)|}{|x-y|^n}dy \le CM(f_2)(x).$$

Therefore, we have that

$$\int_{|x-h|<1/4} |T_{01}f_2(x)|^p\omega(x)dx \le C\int_{|y-h|<5/4} |f(y)|^p\omega(y)dy, \qquad (4.3.4)$$

where C does not depend on h and the coefficients of $P(x,y)$.

Also notice that when $|x-h| < 1/4, |y-h| \ge 5/4$, we have $|x-y| > 1$. Hence

$$T_{01}f_3(x) = 0. \qquad (4.3.5)$$

By (4.3.3), (4.3.4) and (4.3.5), we obtain that

$$\int_{|x-h|<1/4} |T_{01}f(x)|^p\omega(x)dx \le C\int_{|y-h|<5/4} |f(y)|^p\omega(y)dy, \qquad (4.3.6)$$

where C does not depend on h and the coefficients of $P(x,y)$.

Obviously, when $|x-h|<1/4, |x-y|<1$, we have

$$\left|e^{iP(x,y)}-e^{i\left(R(x,y,h)+\sum a_{\alpha\beta}(y-h)^{\alpha+\beta}\right)}\right|\leq C|x-y|.$$

Therefore, when$|x-h|<1/4$, we have that

$$|T_{02}f(x)|\leq C\int_{|x-y|<1}\frac{|f(y)|}{|x-y|^{n-1}}dy\leq CM(f\chi_{B(h,5/4)})(x).$$

It follows that

$$\int_{|x-h|<1/4}|T_{02}f(x)|^p\omega(x)dx\leq C\int_{|y-h|<5/4}|f(y)|^p\omega(y)dy, \tag{4.3.7}$$

where C does not depend on h and the coefficients of $P(x,y)$. From (4.3.6) and (4.3.7), it follows that

$$\int_{|x-h|<1/4}|T_0f(x)|^p\omega(x)dx\leq C\int_{|y-h|<5/4}|f(y)|^p\omega(y)dy$$

holds for every $h\in\mathbb{R}^n$. This is equivalent to having that

$$\|T_0f\|_{p,\omega}\leq C\|f\|_{p,\omega}, \tag{4.3.8}$$

where C does not depend on the coefficients of $P(x,y)$.

For $j\geq 1$, we have

$$|T_jf(x)|\leq\int_{2^{j-1}\leq|x-y|<2^j}\frac{|f(y)|}{|x-y|^n}dy\leq CM(f)(x),$$

where C does not depend on j . By the reverse Hölder inequality, there exists an $\varepsilon>0$ such that $\omega^{1+\varepsilon}\in A_p$. Thus, we have

$$\|T_jf\|_{p,\omega^{1+\varepsilon}}\leq C\|f\|_{p,\omega^{1+\varepsilon}}. \tag{4.3.9}$$

On the other hand, applying a method similar to that of proving (4.1.9), we can get

$$\|T_j\|_2\leq C2^{-j\varepsilon}.$$

And it is easy to see that the norm of T_j on L^1 or L^∞ are uniformly bounded. Thus, we can deduce that

$$\|T_jf\|_p\leq C2^{-j\delta}\|f\|_p, \tag{4.3.10}$$

where C only depends on the total degree of $P(x,y)$ and $\delta > 0$. By (4.3.9), (4.3.10) and the Stein-Weiss interpolation theorem with change of measures, we obtain that

$$\|T_j f\|_{p,\omega} \le C2^{-j\theta\delta}\|f\|_{p,\omega}, \tag{4.3.11}$$

where $0 < \theta < 1, \theta$ is independent of j, and C only depends on the degree of $P(x,y)$. It follows from (4.3.8) and (4.3.11) that

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega} \ ,$$

where C depends only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.

Case 2. Assume $\displaystyle\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}| \ne 1$. Denote $b = \left(\displaystyle\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}|\right)^{\frac{1}{k+l}}$, then

$$P(x,y) = \sum_{|\alpha|=k,|\beta|=l} b^{-(k+l)} a_{\alpha\beta}(bx)^{\alpha}(by)^{\beta} + R\left(\frac{bx}{b}, \frac{by}{b}\right) := Q(bx,by).$$

Thus

$$\begin{aligned} Tf(x) &= \text{p.v.} \int_{\mathbb{R}^n} e^{iQ(bx,by)} K(x,y) f(y) dy \\ &= \text{p.v.} \int_{\mathbb{R}^n} e^{iQ(bx,by)} K\left(\frac{bx}{b}, \frac{y}{b}\right) f\left(\frac{y}{b}\right) b^{-n} dy \\ &:= b^{-n} T_b\left(f\left(\frac{\cdot}{b}\right)\right)(bx), \end{aligned}$$

where

$$T_b f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iQ(x,y)} K_b(x,y) f(y) dy$$

and

$$K_b(x,y) = K\left(\frac{x}{b}, \frac{y}{b}\right).$$

It is not difficult to check that K_b satisfies (a) and (b), and then the operator

$$f \to \text{p.v.} \int_{\mathbb{R}^n} K_b(x,y) f(y) dy$$

is of type (L^2, L^2). The conclusion in the case 1 yields that

$$\|T_b g\|_{p,\omega} \le b^n C\|g\|_{p,\omega},$$

where C only depends on the total degree of $P(x,y)$. Noting that $\omega\left(\frac{\cdot}{b}\right)\in A_p$, and $C\left(\omega\left(\frac{\cdot}{b}\right)\right)=C(\omega)$, we have that

$$\begin{aligned}\int_{\mathbb{R}^n}|Tf(x)|^p\,\omega(x)dx &= b^{-np}\int_{\mathbb{R}^n}\left|T_b\left(f\left(\frac{\cdot}{b}\right)\right)(bx)\right|^p\omega(x)dx\\ &= b^{-np}\int_{\mathbb{R}^n}\left|T_b\left(f\left(\frac{\cdot}{b}\right)\right)(x)\right|^p\omega\left(\frac{x}{b}\right)b^{-n}dx\\ &\le C\int_{\mathbb{R}^n}\left|f\left(\frac{x}{b}\right)\right|^p\omega\left(\frac{x}{b}\right)b^{-n}dx\\ &= C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx,\end{aligned}$$

where C depends only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$. ∎

4.4 Multilinear oscillatory singular integrals with rough kernels

In this section, we will study a class of multilinear oscillatory singular integrals that is a generalization of commutators generated by oscillatory singular integrals and a BMO function. Precisely, The multilinear oscillatory singular integral operators T_A is defined by

$$T_Af(x)=\text{p.v.}\int_{\mathbb{R}^n}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}R_{m+1}(A;x,y)f(y)dy, \tag{4.4.1}$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n\times\mathbb{R}^n$, $R_{m+1}(A;x,y)$ denotes the $m+1$-th order Taylor series remainder of A at x expanded about y, more precisely,

$$R_{m+1}(A;x,y)=A(x)-\sum_{|\alpha|\le m}\frac{1}{\alpha!}D^\alpha A(y)(x-y)^\alpha.$$

It is obvious to see that when $m=0$, $R_1(A;x,y)=A(x)-A(y)$. Thus we obtain that

$$T_Af=AT(f)-T(Af)=[A,T]f.$$

In this case, T_A is just a commutator;

The truncated operator corresponding to T_A is defined by

$$S_Af(x)=\text{p.v.}\int_{|x-y|<1}\frac{\Omega(x-y)}{|x-y|^{n+m}}R_{m+1}(A;x,y)f(y)dy, \tag{4.4.2}$$

In this section, we will investigate a class of multilinear oscillatory singular integral operators with rough kernels and their criterion on L^p-Boundedness.

Suppose that Ω satisfies conditions

(1) Ω is homogeneous of degree 0;
(2) $\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0$; and
(3) $\Omega \in L^q(\mathbb{S}^{n-1})$, for some q with $1 < q \leq \infty$.

Theorem 4.4.1 *Suppose that Ω satisfies (1), (3) and a function A has derivatives of order $m+1$-th in* $\mathrm{BMO}(\mathbb{R}^n)$. *The operator T_A defined by (4.4.1) is bounded on $L^p(\mathbb{R}^n)$ if and only if the truncated operator S_A defined by (4.4.2) is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ with the bound depending only on the total degree of $P(x, y)$, but not on the coefficients of $P(x, y)$.*

To prove Theorem 4.4.1, let us first give some lemmas.

Lemma 4.4.1 *Let $b(x)$ be a function on $\mathbb{R}^n$ with derivatives of m-th order in $L^s(\mathbb{R}^n)$ for some s, $n < s \leq \infty$. Then*

$$|R_m(b; x, y)| \leq C_{m,n}|x-y|^m \sum_{|\alpha|=m} \Big(\frac{1}{|I_x^y|}\int_{I_x^y} |D^\alpha b(z)|^s dz\Big)^{1/s},$$

where I_x^y is the cube centered at x, with its sides paralled to the axes , whose diameter is $2\sqrt{n}|x-y|$.

Lemma 4.4.2 *Suppose that Ω satisfies (1), (3) and a function A has derivatives of order m-th in* $\mathrm{BMO}(\mathbb{R}^n)$. *Denote*

$$M_A f(x) = \sup_{r>0} \frac{1}{r^{n+m}} \int_{|x-y|<r} |\Omega(x-y)R_{m+1}(A; x, y)f(y)|\, dy. \qquad (4.4.3)$$

Then M_A is bounded on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$.

Proof. Define $\overline{M}_A$, a variant of M_A, by

$$\overline{M}_A f(x) = \sup_{r>0} \frac{1}{r^{n+m}} \int_{r/2\leq|x-y|<r} |\Omega(x-y)R_{m+1}(A; x, y)f(y)|\, dy. \quad (4.4.4)$$

It follows that

$$
\begin{aligned}
&M_A f(x)\\
&= \sup_{r>0} \frac{1}{r^{n+m}} \int_{|x-y|<r} |\Omega(x-y) R_{m+1}(A;x,y) f(y)|\, dy\\
&\le \sup_{r>0} \sum_{k=0}^{\infty} \frac{1}{r^{n+m}} \int_{r/2^{k+1}\le|x-y|<r/2^k} |\Omega(x-y) R_{m+1}(A;x,y) f(y)|\, dy\\
&= \sum_{k=0}^{\infty} \sup_{r>0} \frac{2^{-k(n+m)}}{\left(\frac{r}{2^k}\right)^{n+m}} \int_{r/2^{k+1}\le|x-y|<r/2^k} |\Omega(x-y) R_{m+1}(A;x,y) f(y)|\, dy\\
&= \sum_{k=0}^{\infty} \frac{1}{2^{k(n+m)}} \sup_{r>0} \frac{1}{\left(\frac{r}{2^k}\right)^{n+m}} \int_{r/2^{k+1}\le|x-y|<r/2^k} |\Omega(x-y) R_{m+1}(A;x,y) f(y)|\, dy\\
&= C\overline{M}_A f(x).
\end{aligned}
$$

Hence if $\overline{M}_A$ is bounded on $L^p(\mathbb{R}^n)$ with $1<p<\infty$, then we immediately obtain that M_A is also bounded on $L^p(\mathbb{R}^n)$ with $1<p<\infty$.

Therefore it suffices to prove Lemma 4.4.2 for $\overline{M}_A$. For fixed $x\in\mathbb{R}^n$, $r>0$, let $Q(x,r)$ be the cube centered at x and have its sidelength $2\sqrt{n}r$. Set

$$
A^Q(y) = A(y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{Q(x,r)}(D^\alpha A) y^\alpha,
$$

where $m_{Q(x,r)}(D^\alpha A)$ denotes the mean value of $D^\alpha A$ on $Q(x,r)$. Clearly we have

$$
R_{m+1}(A^Q;x,y) = R_{m+1}(A;x,y).
$$

In the following, we will consider two cases:

(i) When $m=0$, in this case $A\in \mathrm{BMO}$ and

$$
\begin{aligned}
\overline{M}_A f(x) &= \sup_{r>0} \frac{1}{\mathbb{R}^n} \int_{r/2\le|x-y|<r} |\Omega(x-y)(A(x)-A(y)) f(y)|\, dy\\
&\le \sup_{r>0} \left(\frac{1}{\mathbb{R}^n} \int_{r/2\le|x-y|<r} |\Omega(x-y)|^q\, |f(y)|\, dy \right)^{1/q}\\
&\quad \times \sup_{r>0} \left(\frac{1}{\mathbb{R}^n} \int_{r/2\le|x-y|<r} |A(x)-A(y)|^{q'}\, |f(y)|\, dy \right)^{1/q'}\\
&:= \Big[P_1(f)(x)\Big]^{1/q} \Big[P_2(f)(x)\Big]^{1/q'}.
\end{aligned}
$$

It is easy to check that

$$\|P_1(f)\|_p \le C\|\Omega\|^q_{L^q(\mathbb{S}^{n-1})}\|f\|_p$$

and

$$\|P_2(f)\|_p \le C\|A\|_{\mathrm{BMO}}\|f\|_p.$$

(ii) When $m \ge 1$, We have that

$$\begin{aligned}
&\overline{M}_A f(x)\\
&\le \sup_{r>0}\frac{1}{r^{n+m}}\int_{r/2<|x-y|<r}|\Omega(x-y)R_m(A^Q;x,y)f(y)|dy\\
&\quad+\sup_{r>0}\frac{1}{r^{n+m}}\int_{r/2<|x-y|<r}\left|\Omega(x-y)\sum_{|\alpha|=m}\frac{1}{\alpha!}D^\alpha A^Q(y)(x-y)^\alpha f(y)\right|dy\\
&:=\overline{M}^1_A f(x)+\overline{M}^2_A f(x).
\end{aligned}$$

From Lemma 4.4.1, it follows that

$$\begin{aligned}
\overline{M}^1_A f(x) &\le C\sup_{r>0}\frac{1}{r^n}\int_{r/2<|x-y|<r}\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}|f(y)|dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\cdot M(f)(x),
\end{aligned}$$

where $M(f)(x)$ denotes the Hardy-Littlewood maximal function of f. Then from the L^p- boundedness of Hardy-Littlewood maximal function, it follows that

$$\|\overline{M}^1_A(f)\|_p \le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p.$$

For $1<p<\infty$, we choose a t such that $1<t<p$. For $\overline{M}^2_A$, we have that

$$\begin{aligned}
\overline{M}^2_A f(x) &\le C\sup_{r>0}\frac{1}{\mathbb{R}^n}\int_{r/2<|x-y|<r}\sum_{|\alpha|=m-1}|D^\alpha A(y)-m_{Q(x,r)}(D^\alpha A)||f(y)|dy\\
&\le C\sum_{|\alpha|=m}\left(\sup_{r>0}\frac{1}{\mathbb{R}^n}\int_{r/2<|x-y|<r}|D^\alpha A(y)-m_{Q(x,r)}(D^\alpha A)|^{t'}dy\right)^{1/t'}\\
&\quad\times\sup_{r>0}\left(r^{-n}\int_{|x-y|<r}|f(y)|^t dy\right)^{1/t}\\
&\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}M_t(f)(x),
\end{aligned}$$

where

$$M_t(f)(x) = \Big[M(|f|^t)(x)\Big]^{\frac{1}{t}}.$$

Thus we conclude that

$$\begin{aligned}
\|M_A^2(f)\|_p &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|M_t(f)\|_p \\
&= C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \left[\left(\int_{\mathbb{R}^n} [M(|f|^t)(x)]^{p/t} dx\right)^{t/p}\right]^{1/t} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p.
\end{aligned}$$

Therefore, for $1 < p < \infty$, we have that

$$\|\overline{M}_A f\|_p \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p.$$

■

Proof of Theorem 4.4.1. Observe that if the operator T defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A; x, y) f(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$, then the truncated operator S_A defined by (4.4.2) is also bounded on $L^p(\mathbb{R}^n)$. By dilation invariance, we only treat the case of

$$\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} = 1.$$

Proof of sufficiency. Let k and l be positive integers. Let $P(x,y)$ be a non-trivial real-valued polynomial with degree k in x and l in y. Write

$$P(x,y) = \sum_{|\alpha|\le k, |\beta|\le l} a_{\alpha,\beta} x^\alpha y^\beta$$

and decompose T_A into

$$\begin{aligned}
T_A f(x) &= \int_{|x-y|<1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A; x, y) f(y) dy \\
&\quad + \int_{|x-y|\ge 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A; x, y) f(y) dy \\
&:= T_A^0 f(x) + T_A^\infty f(x).
\end{aligned}$$

Let us now turn our attention to the operator T_A^∞. For this purpose, we write

$$T_A^\infty f = \sum_{d=1}^{\infty} T_A^d f,$$

where

$$T_A^d f(x) = \int_{2^{d-1} \le |x-y| < 2^d} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y) dy.$$

Let us first consider the operator T_A^d, $d \ge 1$. Since

$$\begin{aligned} |T_A^d f(x)| &\le \int_{2^{d-1} \le |x-y| < 2^d} \left| \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y) \right| dy \\ &\le \sup_{r>0} \frac{2^m}{r^{m+n}} \int_{|x-y|<r} |\Omega(x,y) R_{m+1}(A;x,y) f(y)| \, dy = 2^m M_A f(x). \end{aligned}$$

From Lemma 4.4.2, it follows that

$$\|T_A^d f\|_p \le C \|M_A(f)\|_p \le C \|f\|_p, \tag{4.4.5}$$

where C is independent of d, $1 < p < \infty$.

On the other hand, if we can prove that

$$\|T_A^d f\|_2 \le C 2^{-\theta_1 d} \|f\|_2, \tag{4.4.6}$$

then by interpolating between (4.4.5) and (4.4.6), then we will obtain that

$$\|T_A^d f\|_p \le C 2^{-\theta_2 d} \|f\|_p, \qquad 1 < p < \infty, \tag{4.4.7}$$

where θ_1,θ_2 and C are independent of d and f.

The remainder is to prove (4.4.6). To do this, we define

$$\overline{T}_A^d f(x) = \int_{1 \le |x-y| < 2} e^{iP(2^{d-1}x, 2^{d-1}y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y) dy.$$

We first prove that

$$\left\| \overline{T}_A^d f \right\|_2 \le C 2^{-\theta d} \|f\|_2. \tag{4.4.8}$$

Decompose $\mathbb{R}^n$ into $\mathbb{R}^n = \bigcup I_i$, where each I_i is a cube with side of length 1, and the cubes have disjoint interiors. Set $f_i = f\chi_i$. From the definition of $\overline{T}_A^d f$ and f_i, it easily follows that the support of $\overline{T}_A^d f_i$ is contained in a fixed multiple of I_i. Since the supports of the various terms $\overline{T}^d_A f_i$ are bounded overlapping, we have the "almost orthogonality" property

$$\left\|\overline{T}^d_A f\right\|_2 \le C\sum_i \left\|\overline{T}^d_A f_i\right\|_2.$$

Therefore, it suffices to show that there exists a $\theta > 0$ independent of d and f_i (for all i) such that

$$\left\|\overline{T}^d_A f_i\right\|_2 \le C2^{-\theta d}\|f_i\|_2. \tag{4.4.9}$$

For fixed i, denote $\overline{I}_i = 100nI_i$. Let $\phi_i(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \phi_i \le 1$ and ϕ_i is identically 1 on $10\sqrt{n}I_i$ and vanishes outside of $50\sqrt{n}I_i$, $\|D^\gamma \phi_i\| \le C_\gamma$ for all multi-index γ. Let x_0 be a point on the boundary of $80\sqrt{n}I_i$. Denote

$$A^{\phi_i}(y) = R_m\left(A(\cdot) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{\overline{I}_i}(D^\alpha A)(\cdot)^\alpha; y, x_0\right)\phi_i(y).$$

A simple computation shows that $R_{m+1}(A; x, y) = R_{m+1}(A^{\phi_i}; x, y)$. For multi-index α, define

$$\overline{T}^{d,\alpha}_A f(x) = \int_{1\le|x-y|<2} e^{iP(2^{d-1}x, 2^{d-1}y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}(x-y)^\alpha f(y)dy.$$

It follows that

$$\begin{aligned}
\overline{T}^d_A f_i(x) &= \overline{T}^d_{A^{\phi_i}} f_i(x)\\
&= \int_{1\le|x-y|<2} e^{iP(2^{d-1}x, 2^{d-1}y)}\frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A^{\phi_i}; x, y) f_i(y)dy\\
&= A^{\phi_i}(x)\overline{T}^{d,0}_A f_i(x) - \sum_{|\alpha|<m}\frac{1}{\alpha!}\overline{T}^{d,\alpha}_A\left(D^\alpha A^{\phi_i} f_i\right)(x)\\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\overline{T}^{d,\alpha}_A\left(D^\alpha A^{\phi_i} f_i\right)(x)\\
&:= I + II + III.
\end{aligned}$$

Now we will give the estimates of I, II and III respectively. For multi-index β, $|\beta| < m$, we have

$$\begin{aligned}
&D^\beta A^{\phi_i}(y)\\
&= \sum_{\beta=\mu+\nu} C_{\mu,\nu} R_{m-|\mu|}\left[D^\mu\left(A(\cdot) - \sum_{|\alpha|=m}\frac{1}{\alpha!} m_{\overline{I}_i}(D^\alpha A)(\cdot)^\alpha\right); y, x_0\right] D^\nu \phi_i(y).
\end{aligned}$$

By Lemma 4.4.1, we have $|D^\beta A^{\phi_i}(y)| \le C$. For the operator $\overline{T}_A^{d,\alpha}$ with multi-index α, it is not difficult to obtain the following estimate

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_p \le C2^{-(\theta+m-|\alpha|)d}\|f\|_p, \quad 1 < p < \infty. \tag{4.4.10}$$

Thus it follows from (4.4.10) that

$$\|I\|_2 \le C2^{-\theta d}\|f_i\|_2$$

and

$$\|II\|_2 \le C2^{-\theta d}\|f_i\|_2.$$

It remains to estimate the third term III. For $0 < \gamma < n$, we have

$$\begin{aligned}\left|\overline{T}_A^{d,\alpha} f(x)\right| &= C\int_{1\le|x-y|<2} |\Omega(x-y)f(y)|dy \\ &\le C_\gamma\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\int_{1\le|x-y|<2}\frac{|f(y)|^{q'}}{|x-y|^{n-\gamma}}dy \\ &\le C_\gamma\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\left[I_\gamma(|f|^{q'})(x)\right]^{1/q'},\end{aligned}$$

where I_γ denotes the usual fractional integral of order γ. Choose $p_1 \ge q'$, for $\sigma > 0$, we may take a γ such that $0 < \gamma < nq'/p_1$ and $1/(p_1+\sigma) = 1/p_1 - \gamma/nq'$. By the Hardy-Littlewood-Sobolev theorem, we get

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_{p_1+\sigma} \le C\|f\|_{p_1}. \tag{4.4.11}$$

By (4.4.10), for $|\alpha| = m$, there exists $\delta_1 > 0$ such that

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_p \le C2^{-\delta_1 d}\|f\|_p. \tag{4.4.12}$$

If $p \ge q'$, then we may set $p = p_1$. Interpolating between (4.4.11) and (4.4.12) yields that

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_{p+\rho} \le C2^{-\delta_2 d}\|f\|_p, \tag{4.4.13}$$

where $\delta_2 > 0$ and $0 < \rho \le \sigma$;

If $1 < p < q'$, then interpolating between (4.4.11) and (4.4.12) yields that

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_{p_2+\rho_p} \le C2^{-\delta_2 d}\|f\|_{p_2}, \tag{4.4.14}$$

where $p < p_2 < p_1$ and $0 < \rho_p < \sigma$. It follows from (4.4.13) and (4.4.14) that

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_{p+\rho_p} \le C 2^{-\delta_2 d} \|f\|_p, \tag{4.4.15}$$

where $1 < p < \infty$ and $0 < \rho_p < \sigma$. On the other hand, if $|\beta| = m$, then

$$\begin{aligned}
&D^\beta A^{\phi_i}(y) \\
&= \sum_{\beta=\mu+\nu, |\mu|<m} C_{\mu,\nu} R_{m-|\mu|} \\
&\qquad \times \left[D^\mu \left(A(\cdot) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{\overline{I}_i}(D^\alpha A)(\cdot)^\alpha \right); y, x_0 \right] D^\nu \phi_i(y) \\
&\quad + \sum_{|\alpha|=m} \Big(D^\alpha A(y) - m_{\overline{I}_i}(D^\alpha A) \Big) \phi_i(y).
\end{aligned}$$

Thus, it follows that

$$\left| D^\beta A^{\phi_i}(y) \right| \le C \left(1 + \sum_{|\alpha|=m} \left| D^\alpha A(y) - m_{\overline{I}_i}(D^\alpha A) \right| \right).$$

and this shows for any $t > 1$ that

$$\left\| D^\beta A^{\phi_i} \right\|_t \le C.$$

We choose $\eta > 0$ and $1 < t < \infty$ such that

$$\frac{1}{2} + \frac{1}{t} = \frac{1}{2-\eta}.$$

It follows from (4.4.15) that

$$\begin{aligned}
\|III\|_2 &\le C 2^{-\delta_2 d} \sum_{|\alpha|=m} \left\| D^\alpha A^{\phi_i} f_i \right\|_{2-\eta} \\
&\le C 2^{-\delta_2 d} \sum_{|\alpha|=m} \|D^\alpha A^{\phi_i}\|_t \|f_i\|_2 \\
&\le C 2^{-\delta_2 d} \|f_i\|_2.
\end{aligned}$$

All estimates above imply that (4.4.8) is valid. By the process of proving (4.4.8), it is easy to see that inequality (4.4.8) also holds if $\overline{T}_A^{d,\alpha} f$ is replaced by $T_A^{d,\alpha} f$. Thus, we obtain (4.4.7).

Let us now turn to prove that T_A^0 is also bounded on $L^p(\mathbb{R}^n)$, i.e.,

$$\left\|T_A^0 f\right\|_p \le C\|f\|_p. \tag{4.4.16}$$

We will carry out an argument by a double induction on the degree in x and y of the polynomial $P(x,y)$. If $P(x,y)$ depends only on x or only on y, it is obvious that there cases are trivial. Let k and l be two positive integers and let $P(x,y)$ have degree k in x and l in y. We assume that (4.4.16) holds for all polynomial which are sums of monomials of degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y. Rewrite

$$P(x,y) = \sum_{|\alpha|=k,|\beta|=l} a_{\alpha,\beta}\left(x^\alpha y^\beta - y^{\alpha+\beta}\right) + P_0(x,y).$$

Obviously $P_0(x,y)$ satisfies the inductive hypothesis. It follows that

$$\begin{aligned}
T_A^0 f(x) &= \int_{|x-y|<1} e^{iP_0(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y)dy \\
&\quad + \int_{|x-y|<1} \left(e^{iP(x,y)} - e^{iP_0(x,y)}\right) \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y)dy \\
&:= T_A^{01} f(x) + T_A^{02} f(x).
\end{aligned}$$

By the inductive hypothesis, we conclude that the operator T_A^{01} is bounded on $L^p(\mathbb{R}^n)$. A direct computation shows that

$$\left|e^{iP(x,y)} - e^{iP_0(x,y)}\right| \le C|x-y|.$$

Denote $\tilde{f}(y) = f(y)\chi_{\{|y|\le 2\}}$. When $|x|\le 1$, we have that

$$\begin{aligned}
\left|T_A^{02} f(x)\right| &\le C\int_{|x-y|<1} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_{m+1}(A;x,y)||\tilde{f}(y)|dy \\
&\le CM_A\tilde{f}(x).
\end{aligned}$$

By Lemma 4.4.2, we obtain

$$\int_{|x|\le 1} |T_A^{02} f(x)|^p dx \le C\int_{|x|\le 2} |f(x)|^p dx.$$

Therefore, for any $h\in\mathbb{R}^n$, it follows that

$$\int_{|x-h|\le 1} |T_A^{02} f(x)|^p dx \le C\int_{|x-h|\le 2} |f(x)|^p dx,$$

where C is independent of h. Integrating the last inequality with respect to h yields our desired result.

Proof of necessity. To do this, we choose $Q(x,y)$ such that it has the property $\mathscr{P}$ and decomposes

$$\begin{aligned}T_Af(x) &= \int_{|x-y|<1} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y)f(y)dy \\ &\quad + \int_{|x-y|\geq 1} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y)f(y)dy \\ &:= T_A^0 f(x) + T_A^\infty f(x).\end{aligned}$$

The previous proof implies that T_A^∞ is bounded on $L^p(\mathbb{R}^n)$, so is T_A^0. It is easy to check that

$$\left(\int_{|x-h|\leq 1} |T_A^0 f(x)|^p dx\right)^{1/p} \leq C\left(\int_{|x-h|\leq 2} |f(x)|^p dx\right)^{1/p},$$

where C is independent of h, $1<p<\infty$. Since $Q(x,y)$ has the property $\mathscr{P}$, when $|x-h|<1$, we have that

$$\begin{aligned}&S_Af(x) \\ &= \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y)f(y)\chi_{B_2(h)}(y)dy \\ &= e^{-iR_0(x,h)} \int_{|x-y|<1} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} e^{-iQ(x-h,y-h)} e^{-iR_1(y,h)} f(y)\chi_{B_2(h)}(y)dy.\end{aligned}$$

Observe that the Taylor expansion of $e^{iQ(x-h,y-h)}$ is

$$\begin{aligned}e^{-iQ(x-h,y-h)} &= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \left(\sum_{\alpha,\beta} a_{\alpha\beta}(x-h)^\alpha (y-h)^\beta\right)^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} a_{m,\mu,\nu}(x-h)^\mu (y-h)^\nu,\end{aligned}$$

where u and v are multi-indice. Thus, if we set $a=(1,1,\cdots,1)$ and $b=$

$(2, 2, \cdots, 2)$, then we conclude that

$$\begin{aligned}
&\left(\int_{|x-h|<1} |Sf(x)|^p dx\right)^{1/p} \\
&\le \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} |a_{m,\mu,\nu}| \Big(\int_{|x-h|<1} |(x-h)^\mu|^p \\
&\quad \times \left| T_A^0 \left(e^{-iR_1(\cdot,h)} f(\cdot) \chi_{B(h,2)}(\cdot)(\cdot - h)^\nu \right)(x) \right|^p dx \Big)^{1/p} \\
&\le \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} |a_{m,\mu,\nu}| a^\mu \left(\int_{|y-h|<2} |f(y)|^p |(y-h)^\nu|^p dy \right)^{1/p} \\
&\le \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu} |a_{m,\mu,\nu}| a^\mu b^\nu \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p} \\
&\le C \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta \right)^m \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p} \\
&\le C \exp \left(\sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta \right) \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p}.
\end{aligned}$$

Thus

$$\|S_A f\|_p \le C \|f\|_p.$$

4.5 Multilinear oscillatory singular integrals with standard kernels

Suppose that $K(x, y)$ is a Calderón-Zygmund standard kernel (also called distribution kernel). That means, $K(x, y)$ satisfies

$$(a) \quad |K(x,y)| \le \frac{C}{|x-y|^n}, \quad x \ne y,$$

$$(b) \quad |\nabla_x K(x,y)| + |\nabla_y K(x,y)| \le \frac{C}{|x-y|^{n+1}}, \quad x \ne y.$$

Obviously the kernel K does not have homogeneity.

Let us study in this section an oscillatory singular integral operator defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A; x, y) f(y) dy, \qquad (4.5.1)$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$.

Theorem 4.5.1 *Suppose that $K(x,y)$ satisfies (a) and (b). If A has derivatives of order m in BMO $(\mathbb{R}^n)$, then for $1 < p < \infty$, then the following two facts are equivalent:*

(i) If $P(x,y)$ is a non-trivial real valued polynomial, then T_A is bounded from L^p to L^p with the bound $C(degP, n)\left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}\right)$, where deg P denotes the total degree of the polynomial $P(x,y)$.

(ii) The truncated operator

$$S_A f(x) = p.v. \int_{|x-y|<1} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A; x, y) f(y) dy$$

is bounded from L^p to L^p with the bound $C\left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}\right)$.

Proof of Theorem 4.5.1. By dilation invariance, we only deal with the case

$$\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} = 1.$$

(ii)⇒(i). Let k and l be positive integers. Let $P(x,y)$ be a non-trivial real-valued polynomial with degree k in x and l in y. Write $P(x,y) = \sum_{|\alpha|\le k, |\beta| \le l} a_{\alpha,\beta} x^\alpha y^\beta$ and decompose T_A into

$$\begin{aligned} T_A f(x) = & \int_{|x-y|<1} e^{iP(x,y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A; x, y) f(y) dy \\ & + \int_{|x-y|\ge 1} e^{iP(x,y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A; x, y) f(y) dy \\ & := T_A^0 f(x) + T_A^\infty f(x). \end{aligned}$$

We can easily obtain

$$\|T_A^0 f\|_p \le C \|f\|_p \ .$$

Let us now turn our attention to the operator T_A^∞. For this purpose we write

$$T_A^\infty f(x) = \sum_{d=1}^{\infty} T_A^d f(x) \ ,$$

where

$$T_A^d f(x) = \int_{2^{d-1}\le|x-y|<2^d} e^{iP(x,y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y)f(y)dy.$$

We first consider the operator T_A^d, $d \ge 1$. Since

$$\begin{aligned} |T_A^d f(x)| &\le \int_{2^{d-1}\le|x-y|<2^d} \left| \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y)f(y) \right| dy \\ &\le \sup_{r>0} \frac{2^m}{r^m} \int_{|x-y|<r} |K(x,y)R_{m+1}(A;x,y)f(y)|\, dy := M_A f(x), \end{aligned}$$

it follows from Lemma 4.4.2 that

$$\|T_A^d f\|_p \le C\|M_A(f)\|_p \le C\|f\|_p, \tag{4.5.2}$$

where C is independent of d, $1 < p < \infty$. On the other hand, if we can prove that

$$\|T_A^d f\|_2 \le C 2^{-\theta_1 d}\|f\|_2, \tag{4.5.3}$$

then by interpolating between (4.5.2) and (4.5.3), we will obtain that

$$\|T_A^d f\|_p \le C 2^{-\theta_2 d}\|f\|_p, \qquad 1 < p < \infty, \tag{4.5.4}$$

where θ_1,θ_2 and C are independent of d and f.

The remainder is to prove (4.5.3). To do this, we define

$$\overline{T}_A^d f(x) = \int_{1\le|x-y|<2} e^{iP(2^{d-1}x,2^{d-1}y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y)f(y)dy.$$

We shall prove that

$$\|\overline{T}_A^d f\|_2 \le C 2^{-\theta d}\|f\|_2. \tag{4.5.5}$$

Decompose $\mathbb{R}^n$ into $\mathbb{R}^n = \bigcup I_i$, where I_i is a cube with side of length 1, and the cubes have disjoint interiors. Set $f_i = f\chi_i$. From the definition of $\overline{T}_A^d f$ and f_i, it easily follows that the support of $\overline{T}_A^d f_i$ is contained in a fixed multiple of I_i. Since the supports of the various terms $\overline{T}_A^d f_i$ have bounded overlaps, we have the "almost orthogonality" property

$$\|\overline{T}_A^d f\|_2 \le C \sum_i \|\overline{T}_A^d f_i\|_2.$$

Therefore, it suffices to show that there exists a $\theta > 0$ independent of d and f_i (for all i) such that

$$\|\overline{T}_A^d f_i\|_2 \le C 2^{-\theta d}\|f_i\|_2. \tag{4.5.6}$$

For fixed i, denote $\overline{I}_i = 100nI_i$. Let $\phi_i(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \phi_i \le 1$ and ϕ_i is identically 1 on $10\sqrt{n}I_i$ and vanishes outside of $50\sqrt{n}I_i$, $\|D^\gamma \phi_i\|_\infty \le C_\gamma$ for all multi-index γ. Let x_0 be a point on the boundary of $80\sqrt{n}I_i$. Denote

$$A^{\phi_i}(y) = R_m\left(A(\cdot) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{\overline{I}_i}(D^\alpha A)(\cdot)^\alpha; y, x_0\right)\phi_i(y).$$

A simple computation shows that $R_{m+1}(A;x,y) = R_{m+1}(A^{\phi_i};x,y)$. For multi-index α, define

$$\overline{T}_A^{d,\alpha} f(x) = \int_{1\le|x-y|<2} e^{iP(2^{d-1}x,2^{d-1}y)} \frac{K(x,y)}{|x-y|^m}(x-y)^\alpha f(y)dy.$$

It follows that

$$\begin{aligned}
\overline{T}_A^d f_i(x) &= \overline{T}_{A^{\phi_i}}^d f_i(x) \\
&= \int_{1\le|x-y|<2} e^{iP(2^{d-1}x,2^{d-1}y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A^{\phi_i};x,y) f_i(y)dy \\
&= A^{\phi_i}(x)\overline{T}_A^{d,0} f_i(x) - \sum_{|\alpha|<m} \frac{1}{\alpha!}\overline{T}_A^{d,\alpha}\left(D^\alpha A^{\phi_i} f_i\right)(x) \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!}\overline{T}_A^{d,\alpha}\left(D^\alpha A^{\phi_i} f_i\right)(x) \\
&:= I + II + III.
\end{aligned}$$

In the following, we will give the estimates of I, II and III respectively. For multi-index β, $|\beta| < m$, we have

$$\begin{aligned}
&D^\beta A^{\phi_i}(y) \\
&= \sum_{\beta=\mu+\nu} C_{\mu,\nu} R_{m-|\mu|}\left[D^\mu\left(A(\cdot) - \sum_{|\alpha|=m}\frac{1}{\alpha!} m_{\overline{I}_i}(D^\alpha A)(\cdot)^\alpha\right); y, x_0\right] D^\nu \phi_i(y).
\end{aligned}$$

By Lemma 4.4.1, we have $|D^\beta A^{\phi_i}(y)| \le C$. For the operator $\overline{T}_A^{d,\alpha}$ with multi-index α, it is not difficult to obtain the following estimate

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_p \le C2^{-(\theta+m-|\alpha|)d}\|f\|_p, \quad 1 < p < \infty. \tag{4.5.7}$$

Thus it follows from (4.5.7) that

$$\|I\|_2 \le C2^{-\theta d}\|f_i\|_2$$

and

$$\|II\|_2 \le C2^{-\theta d}\|f_i\|_2.$$

It remains to estimate the third term III . By the property (i) on $K(x,y)$, for $0<\gamma<n$, we have

$$\begin{aligned}\left|\overline{T}_A^{d,\alpha} f(x)\right| &= C\int_{1\le|x-y|<2}|f(y)|dy\\ &\le C\int_{1\le|x-y|<2}\frac{|f(y)|}{|x-y|^{n-\gamma}}dy\\ &\le CI_\gamma(|f|)(x),\end{aligned}$$

where I_γ denotes the usual fractional integral of order γ. For $\sigma>0$, we may take a γ such that $0<\gamma<n/p$ and $1/(p+\sigma)=1/p-\gamma/n$. By the Hardy-Littlewood-Sobolev theorem, we get

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_{p+\sigma} \le C\|f\|_p. \tag{4.5.8}$$

By (4.5.7), for $|\alpha|=m$, there exists $\delta_1>0$ such that

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_{p} \le C2^{-\delta_1 d}\|f\|_p. \tag{4.5.9}$$

Interpolating between inequalities (4.5.8) and (4.5.9) yields that

$$\left\|\overline{T}_A^{d,\alpha} f\right\|_{p+\rho} \le C2^{-\delta_2 d}\|f\|_p, \tag{4.5.10}$$

where $\delta_2>0$ and $0<\rho\le\sigma$. On the other hand, if $|\beta|=m$, then

$$\begin{aligned}&D^\beta A^{\phi_i}(y)\\ &=\sum_{\beta=\mu+\nu,|\mu|<m} C_{\mu,\nu}R_{m-|\mu|}\\ &\quad\times\left[D^\mu\left(A(\cdot)-\sum_{|\alpha|=m}\frac{1}{\alpha!}m_{\overline{I}_i}(D^\alpha A)(\cdot)^\alpha\right);y,x_0\right]D^\nu\phi_i(y)\\ &\quad+\sum_{|\alpha|=m}\Big(D^\alpha A(y)-m_{\overline{I}_i}(D^\alpha A)\Big)\phi_i(y).\end{aligned}$$

Thus, it follows that

$$\left|D^\beta A^{\phi_i}(y)\right| \le C\left(1+\sum_{|\alpha|=m}\left|D^\alpha A(y)-m_{\overline{I}_i}(D^\alpha A)\right|\right).$$

And this shows for any $t > 1$ that

$$\|D^\beta A^{\phi_i}\|_t \le C$$

We choose $\eta > 0$ and $1 < t < \infty$ such that

$$\frac{1}{2} + \frac{1}{t} = \frac{1}{2-\eta}.$$

It follows from (4.5.10) that

$$\begin{aligned}\|III\|_2 &\le C2^{-\delta_2 d} \sum_{|\alpha|=m} \left\|D^\alpha A^{\phi_i} f_i\right\|_{2-\eta}\\ &\le C2^{-\delta_2 d} \sum_{|\alpha|=m} \|D^\alpha A^{\phi_i}\|_t \|f_i\|_2\\ &\le C2^{-\delta_2 d}\|f_i\|_2.\end{aligned}$$

All estimates above imply that (4.5.5) is true. By the process of proving (4.5.5), it is easy to see that inequality (4.5.5) also holds if $\overline{T}_A^{d,\alpha} f$ is replaced by $T_A^{d,\alpha} f$. Thus, we obtain (4.5.3).

(ii)$\Rightarrow$(i). To do this, we choose $Q(x,y)$ such that it has the property $\mathscr{P}$ and decomposes

$$\begin{aligned}T_A f(x) = &\int_{|x-y|<1} e^{iQ(x,y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y) f(y)dy\\ &+ \int_{|x-y|\ge 1} e^{iQ(x,y)} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y) f(y)dy\\ := &T_A^0 f(x) + T_A^\infty f(x).\end{aligned}$$

The previous proof has proved that T_A^∞ is bounded on $L^p(\mathbb{R}^n)$. Thus T_A^0 is also bounded on $L^p(\mathbb{R}^n)$. It is easy to check that

$$\left(\int_{|x-h|\le 1} |T_A^0 f(x)|^p dx\right)^{1/p} \le C \left(\int_{|x-h|\le 2} |f(x)|^p dx\right)^{1/p},$$

where C is independent of h, $1 < p < \infty$. Since $Q(x,y)$ has the property $\mathscr{P}$, when $|x-h| < 1$, it follows that

$$\begin{aligned}&S_A f(x)\\ &= \int_{|x-y|<1} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y) f(y) \chi_{B_2(h)}(y)dy\\ &= e^{-iR_0(x,h)} \int_{|x-y|<1} e^{iQ(x,y)} K(x-y) e^{-iQ(x-h,y-h)} e^{-iR_1(y,h)} f(y) \chi_{B_2(h)}(y)dy.\end{aligned}$$

Observe that the Taylor expansion of $e^{iQ(x-h,y-h)}$ is

$$e^{-iQ(x-h,y-h)} = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \left(\sum_{\alpha,\beta} a_{\alpha\beta}(x-h)^\alpha (y-h)^\beta \right)^m$$
$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} a_{m,\mu,\nu}(x-h)^\mu (y-h)^\nu,$$

where u and v are multi-indice. Thus, if we set $a = (1, 1, \cdots, 1)$ and $b = (2, 2, \cdots, 2)$, then we conclude that

$$\begin{aligned}
&\left(\int_{|x-h|<1} |Sf(x)|^p dx \right)^{1/p} \\
&\leq \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} |a_{m,\mu,\nu}| \\
&\quad \times \left(\int_{|x-h|<1} |(x-h)^\mu|^p \left| T_A^0 \left(e^{-iR_1(\cdot,h)} f(\cdot) \chi_{B(h,2)}(\cdot)(\cdot - h)^\nu \right)(x) \right|^p dx \right)^{1/p} \\
&\leq \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu,} |a_{m,\mu,\nu}| a^\mu \left(\int_{|y-h|<2} |f(y)|^p |(y-h)^\nu|^p dy \right)^{1/p} \\
&\leq \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu,\nu} |a_{m,\mu,\nu}| a^\mu b^\nu \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p} \\
&\leq C \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta \right)^m \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p} \\
&\leq C \exp \left(\sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta \right) \left(\int_{|y-h|<2} |f(y)|^p dy \right)^{1/p}.
\end{aligned}$$

Thus

$$\|S_A f\|_p \leq C \|f\|_p,$$

and then we finish the proof of Theorem 4.5.1.

The following theorem is an application of Theorem 4.5.1.

Theorem 4.5.2 *Suppose that $K(x,y)$ satisfies (a) and (b). Let A have derivatives of order m in BMO$(\mathbb{R}^n)$. If the following two operators*

$$\mathscr{K} f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

and

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y) f(y) dy \tag{4.5.11}$$

are all bounded on $L^2(\mathbb{R}^n)$, then T_A defined by (4.5.1) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with the bound $C(degP, n) \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}}$.

Before proving Theorem 4.5.2, let us first introduce a class of auxiliary multilinear oscillatory singular integral operators defined by

$$\mathcal{T}_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{K(x,y)}{|x-y|^m} Q_{m+1}(A;x,y) f(y) dy \tag{4.5.12}$$

and

$$\mathcal{T} f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{K(x,y)}{|x-y|^m} Q_{m+1}(A;x,y) f(y) dy, \tag{4.5.13}$$

where

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-y)^\alpha.$$

Obviously,

$$R_{m+1}(A;x,y) - Q_{m+1}(A;x,y) = \sum_{|\alpha|=m} \frac{1}{\alpha!} (x-y)^\alpha \Big(D^\alpha A(x) - D^\alpha A(y) \Big),$$

where $D^\alpha A \in \text{BMO}(\mathbb{R}^n)$. Thus we have

$$\begin{aligned} &\mathcal{T}_A f(x) - T_A f(x) \\ &= \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{K(x,y)(x-y)^\alpha}{|x-y|^m} \Big(D^\alpha A(x) - D^\alpha A(y) \Big) f(y) dy \\ &= \sum_{|\alpha|=m} \frac{1}{\alpha!} \Big[D^\alpha A, \mathcal{K}_\alpha \Big] f(x), \end{aligned}$$

where

$$\mathcal{K}_\alpha f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{K(x,y)(x-y)^\alpha}{|x-y|^m} f(y) dy$$

and $[D^\alpha A, \mathcal{K}_\alpha]$ is a commutator. Set

$$K_\alpha(x,y) = \frac{K(x,y)(x-y)^\alpha}{|x-y|^m}.$$

It is easy to check that $K_\alpha(x,y)$ satisfies the condition (a) and (b). Define

$$\mathcal{M}_A f(x) = \sup_{r>0} \frac{1}{r^m} \int_{|x-y|<r} \big|K(x,y)Q_{m+1}(A;x,y)f(y)\big| dy.$$

It is not difficult to get that

$$\|M_A f\|_p \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p.$$

To prove Theorem 4.5.2, we need some lemmas.

Lemma 4.5.1 *Suppose that $K(x,y)$ and A be the same as in Theorem 4.5.2. Moreover, $b(x,y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $1 < p < \infty$.*

If the operator

$$T_b f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y) b(x,y) f(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$ with the bound $E \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}$, then the truncated operator

$$S_b f(x) = \text{p.v.} \int_{|x-y|<1} \frac{K(x,y)}{|x-y|^m} R_{m+1}(A;x,y) b(x,y) f(y) dy$$

is also bounded on $L^p(\mathbb{R}^n)$ with the bound $C(E+\|b\|_\infty) \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}\right)$.

If the operator

$$\mathcal{T}_b f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{K(x,y)}{|x-y|^m} Q_{m+1}(A;x,y) b(x,y) f(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$ with the bound $E \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}$, then the truncated operator

$$\mathscr{S}_b f(x) = \text{p.v.} \int_{|x-y|<1} \frac{K(x,y)}{|x-y|^m} Q_{m+1}(A;x,y) b(x,y) f(y) dy$$

is also bounded on $L^p(\mathbb{R}^n)$ with the bound $C(E+\|b\|_\infty) \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}\right)$;

The proof of Lemma 4.5.1 is similar to that of Theorem 4.2.2.

Lemma 4.5.2 *Suppose that $K(x,y)$ and A be the same as in Theorem 4.5.2. If the truncated operator*

$$\mathscr{S}f(x) = \text{p.v.} \int_{|x-y|<1} \frac{K(x,y)}{|x-y|^m} Q_{m+1}(A;x,y) f(y)dy \tag{4.5.14}$$

is bounded on $L^p(\mathbb{R}^n)$ $1<p<\infty$ with the bound

$$C(degP, n)\left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}}\right),$$

then for any non-degenerate polynomial $P(x,y)$, the operator $\mathcal{T}_A$ defined by (4.5.12) is bounded on $L^p(\mathbb{R}^n)$ with the bound

$$C(degP, n)\left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}}\right).$$

The proof of Lemma 4.5.2 is similar to that of Theorem 4.5.1.

Lemma 4.5.3 *Suppose that $K(x,y)$ and A be the same as in Theorem 4.5.2. Let $f \in L^2(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$. For $\varepsilon > 0$, set*

$$I_\varepsilon = \int_{|y|\geq\varepsilon} \frac{K(x_0,y)}{|x_0-y|^m} R_{m+1}(A;x_0,y) f(y)dy,$$

where $x_0 \in \{x : |x| = \frac{3\varepsilon}{5}\}$. If T defined by (4.5.11) is bounded on $L^p(\mathbb{R}^n)$, then

$$\begin{aligned} &\varepsilon^{-n}\int_{|x|\leq\varepsilon/2} |Tf(x) - I_\varepsilon| dx \\ &\leq CM_2 f(0) + \varepsilon^{-n}\int_{|x|\leq\varepsilon/2}\int_{|y|\geq\varepsilon}\Big|\frac{K(y,x)}{|x-y|^m} R_{m+1}(A;x,y) \\ &\quad - \frac{K(x_0,y)}{|x_0-y|^m} R_{m+1}(A;x_0,y)\Big| |f(y)| dydx. \end{aligned}$$

Proof. Define $f_1 = f\chi_{\{|x|<\varepsilon\}}$ and $f_2 = f\chi_{\{|x|\geq\varepsilon\}}$. Then $f = f_1 + f_2$. When $|x| < \varepsilon/2$, we have

$$\begin{aligned}
&|Tf(x) - I_\varepsilon| \\
&\leq |Tf_1(x)| + |Tf_2(x) - I_\varepsilon| \\
&\leq |Tf_1(x)| \\
&\quad + \int_{|y|\geq\varepsilon} \left| \frac{K(x,y)}{|y-x|^m} R_{m+1}(A;x,y) - \frac{K(x_0,y)}{|x_0-y|^m} R_{m+1}(A;y,x_0) \right| |f(y)|dy.
\end{aligned}$$

So, it suffices to prove

$$\int_{|x|\leq\varepsilon/2} |Tf_1(x)|dx \leq C_2\varepsilon^n M_2 f(0).$$

Since T is bounded on $L^2(\mathbb{R}^n)$, using Hölder's inequality implies the desired result. ∎

To prove Theorem 4.5.2, we also need two propositions.

Proposition 4.5.1 *Suppose that $K(x,y)$ and A be the same as in Theorem 4.5.2. If the operator $\mathcal{T}$ defined by (4.5.13) is bounded on $L^2(\mathbb{R}^n)$ with the bound $\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}$, then for any non-degenerate $P(x,y)$, the operator $\mathcal{T}_A$ defined by (4.5.12) is bounded on $L^p(\mathbb{R}^n)$, $1 < p \leq 2$, with the bound $C(degP, n)\left(1 + \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}\right)$.*

Proof. If we can prove that $\mathcal{T}$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, then by the well-known interpolation theorem we obtain that $\mathcal{T}$ is bounded on $L^p(\mathbb{R}^n)(1 < p \leq 2)$. Lemma 4.5.1 directly implies that the truncated operator $\mathscr{S}$ defined by (4.5.14) is also bounded on $L^p(\mathbb{R}^n)(1 < p \leq 2)$. Lemma 4.5.2 shows us that the L^p- boundedness of $\mathcal{T}_A$ immediately follows from L^p- boundedness of $\mathscr{S}$.

Let us now prove that $\mathcal{T}$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. By a standard argument, it suffices to show that there exists a constant C such that

$$\int_{\mathbb{R}^n} |\mathcal{T}a(x)|dx \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}$$

for each $H^1(\mathbb{R}^n)$-atom a, where C is independent of a. For this purpose, suppose that $\mathrm{supp}(a) \subset B$ for some ball B. Then

$$\int_{\mathbb{R}^n} |\mathcal{T}a(x)|dx = \int_{2B} |\mathcal{T}a(x)|dx + \int_{\mathbb{R}^n\setminus 2B} |\mathcal{T}a(x)|dx := I + II.$$

The estimate for I follows from the boundedness of $\mathcal{T}$ on $L^2(\mathbb{R}^n)$.

$$\begin{aligned}
I &= \int_{2B} |\mathcal{T}a(x)|dx \\
&\le \left(\int_{2B} |\mathcal{T}a(x)|^2 dx\right)^{1/2} \left(\int_{2B} dx\right)^{1/2} \\
&\le C|B|^{1/2} \cdot \|a\|_2 \le C.
\end{aligned}$$

To estimate II we set $\tilde{B} = 1.2B, B^* = 5\sqrt{n}B$. Let y_0 be on the boundary of $\tilde{B}$, $m_{B^*}(b)$ denotes the mean value of b on B^*. Set

$$A^{B^*}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{B^*}(D^\alpha A)x^\alpha.$$

Obviously, $Q_{m+1}(A, x, y) = Q_{m+1}(A^{B^*}, x, y)$, when $x \in \mathbb{R}^n \setminus 2B$,

$$\begin{aligned}
\mathcal{T}a(x) &= \int_B \frac{K(x,y)}{|x-y|^m} Q_{m+1}(A^{B^*}; x, y)a(y)dy \\
&= \int_B \frac{K(x,y)}{|x-y|^m} R_m(A^{B^*}; x, y)a(y)dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_B \frac{K(x,y)(x-y)^\alpha}{|x-y|^m} D^\alpha A^{B^*}(x)a(y)dy \\
&:= \mathcal{T}^1 a(x) + \mathcal{T}^2 a(x).
\end{aligned}$$

For $\mathcal{T}^1 a(x)$, when $x \in \mathbb{R}^n \setminus 2B$,

$$\begin{aligned}
\left|\mathcal{T}^1 a(x)\right| &= \left| \int_B \left(\frac{K(x,y)}{|x-y|^m} R_m(A^{B^*}; x, y) - \frac{K(x,y_0)}{|x-y_0|^m} R_m(A^{B^*}; x, y_0) \right) a(y)dy \right| \\
&\le C|x-y_0|^{-n-m} \int_B \left| R_m(A^{B^*}; x, y) - R_m(A^{B^*}; x, y_0) \right| |a(y)|dy \\
&\quad + C \int_B \left| \frac{K(x,y)}{|x-y|^m} - \frac{K(x,y_0)}{|x-y_0|^m} \right| \left| R_m(A^{B^*}; x, y_0)a(y) \right| dy.
\end{aligned}$$

By the definition of $R_m(A^{B^*}; x, y)$, we have

$$R_m(A^{B^*}; x, y) - R_m(A^{B^*}; x, y_0) = \sum_{|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta A^{B^*}; y_0, y)(x-y_0)^\beta.$$

For $|\beta| < m$, applying Lemma 4.4.1 leads to

$$\left| R_{m-|\beta|}(D^\beta A^{B^*}; y_0, y) \right| \le C|y-y_0|^{m-|\beta|} \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}.$$

Thus

$$\begin{aligned}&\left|R_m(A^{B^*};x,y)-R_m(A^{B^*};x,y_0)\right|\\&\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|y-y_0|^{m-|\beta|}|x-y_0|^{|\beta|}\|D^\alpha A\|_{\mathrm{BMO}}.\end{aligned}$$

Therefore

$$\begin{aligned}&|x-y_0|^{-n-m}\int_B\left|R_m(A^{B^*};x,y)-R_m(A^{B^*};x,y_0)\right||a(y)|dy\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\int_B\frac{|y-y_0||a(y)|}{|x-y|^{n+1}}dy,\end{aligned}\tag{4.5.15}$$

where we use the following relation $|x-y|\approx|x-y_0|\ge|y-y_0|$.

When $x\in\mathbb{R}^n\setminus 2B$, we have that

$$\begin{aligned}&\int_B\left|\frac{K(x,y)}{|x-y|^m}-\frac{K(x,y_0)}{|x-y_0|^m}\right|\left|R_m(A^{B^*};x,y_0)a(y)\right|dy\\&\le C\int_B\frac{|y-y_0|}{|x-y|^{m+n+1}}|R_m(A^{B^*};x,y_0)a(y)|dy\\&=C\sum_{|\alpha|=m}\int_B\frac{|y-y_0|}{|x-y|^{n+1}}\\&\quad\times\left[\left(\frac{1}{|I_x^{y_0}|}\int_{I_x^{y_0}}|D^\alpha A(z)-m_{B^*}(D^\alpha A\right)|^s dz)^{1/s}\right]|a(y)|dy\\&\le C\sum_{|\alpha|=m}\int_B\frac{|y-y_0|}{|x-y|^{n+1}}\\&\quad\times\left[\left(\frac{1}{|C_1I_x^{y_0}|}\int_{C_1I_x^{y_0}}|D^\alpha A(z)-m_{C_1I_x^{y_0}}(D^\alpha A\right)|^s dz)^{1/s}\right]|a(y)|dy\\&\quad+C\sum_{|\alpha|=m}\int_B\frac{|y-y_0|}{|x-y|^{n+1}}\left|m_{C_1I_x^{y_0}}(D^\alpha A)-m_{B^*}(D^\alpha A)\right||a(y)|dy.\end{aligned}$$

Note that $B^*\in C_1I_x^{y_0}$. By a well-known inequality, if $f\in\mathrm{BMO}$, then

$$|m_{B_1}(f)-m_{B_2}(f)|\le C\log\frac{|B_2|}{|B_1|}\|f\|_{\mathrm{BMO}},B_1\subset B_2,2|B_1|\le|B_2|,$$

where B_i is cubes or balls ($i=1,2$), C_1 is a constant. In fact, we may let

$C_1 = 5\sqrt{n}$ and get

$$\begin{aligned} &\int_B \left| \frac{K(x,y)}{|x-y|^m} - \frac{K(x,y_0)}{|x-y_0|^m} \right| \left| R_m(A^{B^*}; x, y_0) a(y) \right| dy \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \log \frac{|x-y_0|}{r_B} \int_B \frac{|y-y_0||a(y)|}{|x-y|^{n+1}} dy, \end{aligned} \tag{4.5.16}$$

where r_B denotes the radius of B. Combining (4.5.15) with (4.5.16) yields that

$$|\mathcal{T}^1 a(x)| \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \log \frac{|x-y_0|}{r_B} \int_B \frac{|y-y_0||a(y)|}{|x-y|^{n+1}} dy.$$

For the estimate of $\mathcal{T}^2 a(x)$, we have that

$$\begin{aligned} &|\mathcal{T}^2 a(x)| \\ &\le \sum_{|\alpha|=m} \frac{1}{\alpha!} \left| D^\alpha A^{B^*}(x) \right| \int_B \left| \frac{K(x,y)(x-y)^\alpha}{|x-y|^m} - \frac{K(x,y)(x-y_0)^\alpha}{|x-y|^m} \right| |a(y)| dy \\ &\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} |D^\alpha A^{B^*}(x)| \int_B \left| \frac{K(x,y)(x-y_0)^\alpha}{|x-y|^m} - \frac{K(x,y_0)(x-y_0)^\alpha}{|x-y_0|^m} \right| |a(y)| dy. \end{aligned}$$

Since $K(x,y)$ and $K_\alpha(x,y)$ satisfy (b), we conclude that

$$|\mathcal{T}^2 a(x)| \le C \sum_{|\alpha|=m} \frac{1}{\alpha!} |D^\alpha A^{B^*}(x)| \int_B \frac{|y-y_0|}{|x-y|^{n+1}} |a(y)| dy.$$

Thus, from the previous estimates it follows that

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus 2B} |\mathcal{T}^1 a(x)| dx \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \int_B |a(y)| dy \sum_{k=1}^\infty \int_{A_k} \frac{|y-y_0|}{|x-y|^{n+1}} \log \frac{|x-y_0|}{r_B} dx, \end{aligned}$$

where $A_k = 2^{k+1}B \setminus 2^k B \quad (k \ge 1)$. Notice that $y \in B$, when $x \in A_k$. Then we have that

$$\frac{|y-y_0|}{|x-y|^{n+1}} \le \frac{C r_B}{(2^k r_B)^{n+1}} = \frac{Ck}{(2^k r_B)^n 2^k}.$$

Thus

$$\int_{A_k} \frac{|y-y_0|}{|x-y|^{n+1}} \log \frac{|x-y_0|}{r_B} dx \le \frac{Ck}{(2^k r_B)^n 2^k} (2^k r_B)^n \le Ck 2^{-k}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^n\setminus 2B}|\mathcal{T}^1a(x)|dx &\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^\infty k2^{-k}\int_B|a(y)|dy\\ &\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus 2B}|\mathcal{T}^2a(x)|dx\\ &\leq C\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n\setminus 2B}\int_B\frac{|y-y_0|}{|x-y|^{n+1}}|D^\alpha A^{B^*}(x)\||a(y)|dydx\\ &\leq C\sum_{|\alpha|=m}\frac{1}{\alpha!}\sum_{k=1}^\infty\int_B|a(y)|dy\int_{A_k}\frac{|D^\alpha A(x)-m_{Q^*}(D^\alpha A)|}{(2^kr_B)^n2^k}dx\\ &\leq C\sum_{|\alpha|=m}\frac{1}{\alpha!}\sum_{k=1}^\infty k2^{-k}\|D^\alpha A\|_{\mathrm{BMO}}\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}.\end{aligned}$$

It follows that

$$II\leq\int_{\mathbb{R}^n\setminus 2B}|\mathcal{T}^2a(x)|dx+\int_{\mathbb{R}^n\setminus 2B}|\mathcal{T}^2a(x)|dx\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}.$$

This finishes the proof of Proposition 4.5.1. ■

Proposition 4.5.2 *Suppose that $K(x,y)$ and A be the same as in Theorem 4.5.2. If the operator T defined by (4.5.11) is bounded on $L^2(\mathbb{R}^n)$ with the bound $\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}$, then for any non-degenerate $P(x,y)$, the operator T_A defined by (4.5.1) is bounded on $L^p(\mathbb{R}^n)$, $2\leq p<\infty$, with the bound $C(degP,n)\Big(1+\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\Big)$.*

Proof. By a method similar to that proving Proposition 4.5.1, it suffices to prove that T is bounded from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. Assume $f\in L^2(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$ and $\varepsilon>0$. By Lemma 4.5.3, we get that

$$\begin{aligned}&\varepsilon^{-n}\int_{|y|\leq\varepsilon/2}|Tf(y)-I_\varepsilon|dy\\ &\leq CM_2f(0)+\varepsilon^{-n}\int_{|y|\leq\varepsilon/2}\int_{|x|\geq\varepsilon}\Big|\frac{K(x,y)}{|x-y|^m}R_{m+1}(A;x,y)\\ &\quad-\frac{K(x,y_0)}{|x-y_0|^m}R_{m+1}(A;x,y_0)\Big||f(x)|dxdy.\end{aligned}$$

Obviously, $M_2(0) \le \|f\|_\infty$. Let $B^\varepsilon = \{x : |x| \le 5\sqrt{n}\varepsilon\}$ and $m_{B^\varepsilon}(b)$ denote the mean value of b on B^ε. Set

$$A^{B^\varepsilon}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{B^\varepsilon}(D^\alpha A) x^\alpha.$$

Using the method in the proof of Proposition 4.5.1 , we can easily obtain that

$$\varepsilon^{-n} \int_{|y|\le\varepsilon/2} |Tf(y) - I_\varepsilon| dy \le \left(1 + \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}\right) \|f\|_\infty.$$

By the translation invariance, we obtain that for any ball B, there exists I_B, such that

$$\sup_B \frac{1}{|B|} \int_B |Tf(x) - I_B| dx \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_\infty.$$

Suppose $f \in L^\infty(\mathbb{R}^n)$. Set $B_j = jB$, $j \in \mathbb{N}$, where $B = \{|x| \le 1\}$. If we can prove that there exist γ_j independent of f such that

$$Tf(x) = \lim_{j\to\infty} \Big(T(f\chi_{B_j})(x) - \gamma_j\Big) \tag{4.5.17}$$

exists, where the convergence is the meaning of local L^1 norm or almost everywhere, then by the definition of space $\mathrm{BMO}(\mathbb{R}^n)$, we will prove that T is bounded from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. Actually, for some $R > 0$, if $|x| \le R$, then we take x_0 with $|x_0| = 6R/5$. Set

$$\gamma_j = \int_{2R\le|y-x_0|\le j} \frac{K(x_0, y)}{|x_0 - y|^m} R_{m+1}(A; x_0, y) f(y) dy,$$

where the integral above exists obviously. Denote

$$g_j(x) = T(f\chi_{B_j})(x) - \gamma_j.$$

When $j, l > 2R + 1$ and $j > l$, we have that

$$g_j(x) = g_l(x) + S_{lj}(x),$$

where

$$\begin{aligned} &S_{lj}(x) \\ &= \int_{l<|y-x_0|\le j} \left(\frac{K(x, y)}{|x - y|^m} R_{m+1}(A; x, y) - \frac{K(x_0, y)}{|x_0 - y|^m} R_{m+1}(A; x_0, y)\right) f(y) dy. \end{aligned}$$

By a well-known inequality on BMO before (4.5.16) and by an argument similar to that of estimating $T^1 a(x)$ above, we obtain that

$$|S_{lj}(x)| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_\infty \int_{l<|y-x_0|\leq j} \frac{|x-x_0|}{|x_0-y|^{n+1}} \log\left(\frac{|y-x_0|}{|x-x_0|}\right) dy.$$

For $|x| \leq R$ and $|x_0| = 6R/5$, when $l, j \to \infty$, a simple computation yields that

$$\int_{l<|y-x_0|\leq j} \frac{|x-x_0|}{|x_0-y|^{n+1}} \log\left(\frac{|y-x_0|}{|x-x_0|}\right) dy$$

converges to 0 in agreement. This means that the limit in (4.5.17) exists. Thus

$$\|Tf\|_{\mathrm{BMO}} \leq \overline{\lim_{j\to\infty}} \|g_j\|_{\mathrm{BMO}} = \overline{\lim_{j\to\infty}} \|T(f\chi_{B_j})\|_{\mathrm{BMO}} \leq C\|f\|_\infty.$$

This finishes the proof of Proposition 4.5.2. ■

Proof of Theorem 4.5.2. We first prove that operator

$$\mathscr{K}_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x,y) f(y) dy$$

is bounded on $L^2(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}^n)$, set

$$f_\alpha = \frac{f(\cdot)(x-\cdot)^\alpha}{|x-\cdot|^m},$$

then $f_\alpha \in L^2(\mathbb{R}^n)$, where $|\alpha| = m$. Thus

$$\begin{aligned} \|\mathscr{K}_\alpha f\|_2 &= \left\| \int_{\mathbb{R}^n} K_\alpha(\cdot,y) f(y) dy \right\|_2 \\ &= \left\| \int_{\mathbb{R}^n} K(\cdot,y) f_\alpha(y) dy \right\|_2 \\ &= \|\mathscr{K} f_\alpha\|_2 \\ &\leq C\|f_\alpha\|_2 \\ &= C\|f\|_2. \end{aligned}$$

Next we will prove that operator $\mathcal{T}$ defined by (4.5.13) is also bounded on $L^2(\mathbb{R}^n)$. Since $K_\alpha(x,y)$ satisfies the condition (a) and (b), and $\mathscr{K}_\alpha$ is bounded on $L^2(\mathbb{R}^n)$, we therefore deduce that $\mathscr{K}_\alpha$ is bounded on $L^p_\omega(\mathbb{R}^n)$, where $\omega \in A_p$ is a weight function and $1 < p < \infty$. Thus for $D^\alpha A \in$

BMO($\mathbb{R}^n$), the corresponding commutator $[D^\alpha A, \mathscr{K}_\alpha]$ is bounded on $L^p(\mathbb{R}^n)$. So

$$\mathscr{K}_{D^m A} = \sum_{|\alpha|=m} \frac{1}{\alpha!}[D^\alpha A, \mathscr{K}_\alpha]$$

is also bounded on $L^p(\mathbb{R}^n)$. Since $\mathcal{T} = T + \mathscr{K}_{D^m A}$ and T is bounded on $L^2(\mathbb{R}^n)$, it follows that $\mathcal{T}$ is bounded on $L^2(\mathbb{R}^n)$. Thus it follows from Proposition 4.5.1 that $\mathcal{T}_A$ is bounded on $L^p(\mathbb{R}^n)$ with $1 < p \le 2$.

From Lemma 4.5.1 with $D^\alpha A$ replacing A and $R_1(D^\alpha A; x, y) = D^\alpha A(x) - D^\alpha A(y)$, it follows that the corresponding truncated operator

$$S_{D^m A} f(x) = \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|x-y|<1} K_\alpha(x,y)\Big(D^\alpha A(x) - D^\alpha A(y)\Big) f(y)dy$$

is also bounded on $L^p(\mathbb{R}^n)$. Hence by Theorem 4.5.1, we have that

$$\sum_{|\alpha|=m} \frac{1}{\alpha!}[D^\alpha A, \mathcal{K}_\alpha]$$

is also bounded on $L^p(\mathbb{R}^n)$. By

$$T_A = \mathcal{T}_A - \sum_{|\alpha|=m} \frac{1}{\alpha!}[D^\alpha A, \mathcal{K}_\alpha],$$

we obtain that T_A is bounded on $L^p(\mathbb{R}^n)$ with $1 < p \le 2$. This together with Proposition 4.5.2 yields our desired result.

4.6 Notes and references

Theorem 4.1.1 is due to Ricci and Stein [RiS]. Early form of Theorem 4.1.1 with $P(x, y)$ being a bilinear form was first obtained by Phong and Stein [PhS1],[PhS2]. And Theorem 4.1.2 is due to Chanillo and Christ [ChaC]. The weighted version of Theorem 4.1.2 was established by Sato [Sa]. The proof of Theorem 4.1.2 is taken from [Sa].

For real Hardy spaces $H^p(\mathbb{R}^n)$ with $0 < p < 1$, Pan [Pa] gave a counter example to show that there exists a non-trivial polynomial $P(x, y)$ such that the corresponding oscillatory singular integral T is unbounded on $H^p(\mathbb{R}^n)$. For the case $p = 1$, we do not know whether T maps $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. As a substitution result for $p = 1$, Lu and Yang [LuYa] showed that T maps

Herz type Hardy spaces into Herz spaces. Moreover, as a development of the result in [LuYa], Li [Li] gave a systematic work in the case $0 < p < 1$.

Theorem 4.2.1 and Theorem 4.2.2 are due to Lu and Zhang [LuZ1]. It should be pointed out that we do not need the condition (2) on the kernel in Theorem 4.2.1, but need it in Theorem 4.2.2. Theorem 4.2.3 and Theorem 4.2.4 are due to Jiang and Lu [JiL]. As we mentioned in [JiL], the condition (4) in Theorem 4.2.4 is sharp in the following sense: (4) can not be replaced by any $\Omega \in L(\log^+ L)^\varepsilon(\mathbb{S}^{n-1})$ with $0 < \varepsilon < 1$. In addition, the weighted version of Theorem 4.2.4 was established by Ojanen [Oj]. Moreover, Lu and Wu [LuW1] proved that the conclusion of Theorem 4.2.4 still holds if the condition (4) in Theorem 4.2.4 is replaced by $\Omega \in B_q^{0,0}(\mathbb{S}^{n-1})$, where $B_q^{0,0}(\mathbb{S}^{n-1})$ is certain block space (see [LuTW]). In addition, Lu and Wu [LuW2] established a result similar to that of Lu and Wu [LuW1] for commutators of oscillatory singular integrals with rough kernels in certain block spaces. It should be also pointed out that whether the conclusion of Theorem 4.2.4 holds if the condition (4) in Theorem 4.2.4 is replaced by $\Omega \in H^1(\mathbb{S}^{n-1})$ is still a problem, where $H^1(\mathbb{S}^{n-1})$ is the Hardy space on the unit sphere. However, Fan and Pan [FaP1] proved that the above conclusion holds when $P(x,y) = P(x-y)$.

Theorem 4.3.1 is due to Ricci and Stein [RiS]. As the weighted version of Theorem 4.3.1, Theorem 4.3.2 and its proof given here are due to Lu and Zhang [LuZ2].

Theorem 4.4.1 is due to Chen, Hu and Lu [CheHL]. Lemma 4.4.1 is due to Cohen and Gosselin [CohG1]. A part idea in the proof of Lemma 4.4.2 comes from [CohG2]. The weighted version of Theorem 4.4.1 was obtained by Ding, Lu and Yang [DiLYa]. There are two results that improved Theorem 4.4.1. One of them was obtained by Chen and Lu [CheL], where the condition (3) on the kernel in Theorem 4.4.1 was replaced by a weaker condition $\Omega \in L(\log^+ L)^2(\mathbb{S}^{n-1})$. Another result was obtained by Lu and Wu [LuW2], where the condition (3) in the theorem was replaced by a weaker condition $\Omega \in B_q^{0,1}(\mathbb{S}^{n-1})$. Here $B_q^{0,1}(\mathbb{S}^{n-1})$ is a block space on the unit sphere (see [LuTW]). The weighted boundedness for high order commutators of oscillatory singular integrals was established by Ding and Lu [DiL2].

Theorem 4.5.1 is due to Lu [Lu2]. As a direct application of Theorem 4.5.1, Theorem 4.5.2 was established by Lu and Yan [LuY].

For results on L^p boundedness of the maximal operator corresponding to oscillator singular integrals and multilinear oscillatory singular integrals, we did not mention them in this book, and refer to Lu and Zhang [LuZ1] and Yan, Meng and Lan [YML] respectively . For other related results on oscillatory singular integrals and multilinear oscillatory singular integrals, we refer to a survey paper by Lu [Lu3].

Chapter 5

Littlewood-Paley Operator

We know that if $f \in L^2(0, 2\pi)$, then by the Parseval equality we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{k\in\mathbb{Z}} \left|\hat{f}(k)\right|^2, \tag{5.0.1}$$

where for $k \in \mathbb{Z}$ and

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

is called the Fourier coefficient of f. Similarly, by the Plancherel equality we have

$$\|f\|_2 = \|\hat{f}\|_2 \tag{5.0.2}$$

for $f \in L^2(\mathbb{R}^n)$, where

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$$

is called Fourier transform of f.

However, for $1 < p < \infty$, $p \neq 2$, (5.0.1) (or (5.0.2)) does not hold for $f \in L^p(0, 2\pi)$ (or $f \in L^p(\mathbb{R}^n)$). The Littlewood-Paley theory, originated in the 1930s and developed in the late 1950s, is a very effective replacement. It has played a very important role in harmonic analysis, complex analysis and PDE.

5.1 Littlewood-Paley g function

For $f \in L^p(\mathbb{R}^n),\ 1 < p < \infty$, we set $u(x,t)$ to be the Poisson integral of f, i.e.,

$$u(x,t) = \int_{\mathbb{R}^n} P_t(y) f(x-y) dy,$$

where

$$P_t(y) = c_n \frac{t}{(t^2 + |y|^2)^{\frac{n+1}{2}}}, \qquad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}.$$

Then the Littlewood-Paley g function of f is defined by

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x,t)|^2 t dt \right)^{1/2},$$

where

$$|\nabla u(x,t)| = \left| \frac{\partial u}{\partial t} \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j} \right|^2.$$

Stein proved that g function can characterize $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$.

Theorem 5.1.1 *Suppose that $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. Then $g(f) \in L^p(\mathbb{R}^n)$, and there exist constants A, B such that*

$$A\|f\|_p \leq \|g(f)\|_p \leq B\|f\|_p \tag{5.1.1}$$

for every $f \in L^p(\mathbb{R}^n)$.

For the proof of this theorem, Stein [St3]. We shall discuss the properties of more general Littlewood-Paley g function.

Definition 5.1.1 *Suppose that $\varphi(x) \in L^1(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n} \varphi(x) dx = 0. \tag{5.1.2}$$

Then the generalized Littlewood-Paley g function g_φ is defined by (it is still called as the Littlewood-Paley g function for convenience)

$$g_\varphi(f) = \left(\int_0^\infty |\varphi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $\varphi_t(x) = \dfrac{1}{t^n} \varphi\left(\dfrac{x}{t}\right)$.

In Section 2.1, we have obtained the following result by the boundedness of vector-valued singular integral operator (see Theorem 2.1.11), i.e.,

Theorem 5.1.2 *Suppose that* $\varphi(x) \in L^1(\mathbb{R}^n)$ *satisfies* (5.1.2) *and the following properties:*

$$|\varphi(x)| \le \frac{C}{(1+|x|)^{n+\alpha}}, \quad x \in \mathbb{R}^n \tag{5.1.3}$$

$$\int_{\mathbb{R}^n} |\varphi(x+h) - \varphi(x)| dx \le C|h|^{\alpha}, \quad h \in \mathbb{R}^n \tag{5.1.4}$$

where C *and* $\alpha > 0$ *are both independent of* x *and* h. *Then* g_φ *is of type* (p,p) $(1 < p < \infty)$ *and of weak type* $(1,1)$.

Next we will show that g_φ can also characterize $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. In other words, we can also obtain (5.1.1) for g_φ.

Theorem 5.1.3 *Suppose that* $\varphi(x) \in L^1(\mathbb{R}^n)$ *satisfies* (5.1.2)-(5.1.4). *Then there exist constants* A *and* B *such that*

$$A\|f\|_p \le \|g_\varphi(f)\|_p \le B\|f\|_p \tag{5.1.5}$$

for all $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Proof. Clearly we merely need to prove the first inequality in (5.1.5). Let us first begin with an identity. For all $f, h \in L^2(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^{n+1}} \varphi_t * f(x) \cdot \varphi_t * \bar{h}(x) \frac{dxdt}{t} = C \int_{\mathbb{R}^n} f(x) \cdot \bar{h}(x) dx. \tag{5.1.6}$$

Apply (2.1.43) and set

$$C = \int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t},$$

where $\xi \in \mathbb{R}^n$ and C is independent of ξ. Then the left side of (5.1.6) is

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} \varphi_t * f(x) \cdot \varphi_t * \bar{h}(x) dx \frac{dt}{t} \\
&= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} |\widehat{\varphi}(t\xi)|^2 \widehat{f}(\xi) \cdot \widehat{\bar{h}}(\xi) d\xi \frac{dt}{t} \\
&= \int_{\mathbb{R}^n} \widehat{f}(\xi) \cdot \widehat{\bar{h}}(\xi) \cdot \lim_{\varepsilon \to 0} \int_{\varepsilon|\xi|}^\infty |\widehat{\varphi}(t\xi')| \frac{dt}{t} d\xi \\
&= C \int_{\mathbb{R}^n} \widehat{f}(\xi) \cdot \widehat{\bar{h}}(\xi) d\xi,
\end{aligned}$$

from which (5.1.6) follows.

Now we turn to the proof of (5.1.5). For $f \in L^2 \bigcap L^p(\mathbb{R}^n)$, we have

$$\|f\|_p = \sup \left| \int_{\mathbb{R}^n} f(x)\bar{h}(x)dx \right|,$$

where the supremum is taken over all $h \in L^2 \bigcap L^{p'}(\mathbb{R}^n)$ with $\|h\|_{p'} \leq 1$. Then (5.1.6) and the second inequality of (5.1.5) yield

$$\begin{aligned}\left| \int_{\mathbb{R}^n} f(x)\bar{h}(x)dx \right| &= C \left| \int_{\mathbb{R}^{n+1}} \varphi_t * f(x) \cdot \varphi_t * \bar{h}(x) \frac{dxdt}{t} \right| \\ &\leq C \int_{\mathbb{R}^n} g_\varphi(f)(x) \cdot g_\varphi(\bar{h})(x)dx \\ &\leq C\|g_\varphi(f)\|_p \cdot \|g_\varphi(\bar{h})\|_{p'} \\ &\leq CB\|g_\varphi(f)\|_p.\end{aligned}$$

Assume that $f \in L^p(\mathbb{R}^n)$. Since $L^2 \bigcap L^p$ is dense in L^p, we can take $\{f_k\} \subset L^2 \bigcap L^p$ such that f_k converges to f in the sense of L^p norm. Hence we also have $g_\varphi(f_k) \to g_\varphi(f)$ $(k \to \infty)$ in L^p and

$$\|f\|_p = \lim_{k\to\infty} \|f_k\|_p \leq C \lim_{k\to\infty} \|g_\varphi(f_k)\|_p = CB\|g_\varphi(f)\|_p.$$

This finishes the proof of Theorem 5.1.3. ∎

In the following, we give the discrete form of Theorem 5.1.3.

Theorem 5.1.4 *Suppose that $\psi \in \mathscr{S}(\mathbb{R}^n)$ and $\psi(0) = 0$. For $j \in \mathbb{Z}$, define an operator S_j by*

$$(S_j f)^\wedge(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi).$$

Then

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \leq C_p\|f\|_p \tag{5.1.7}$$

for $1 < p < \infty$. Moreover, if

$$\sum_j \left|\psi(2^{-j}\xi)\right|^2 = C \tag{5.1.8}$$

for all $\xi \neq 0$, then

$$\|f\|_p \leq C'_p \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p. \tag{5.1.9}$$

Proof. By Theorem 2.1.11 and Remark 2.1.6 we know that (5.1.7) holds. On the other hand, by (5.1.8) we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_j S_j f(x) \cdot S_j \bar{h}(x) dx \\
&= \sum_j \int_{\mathbb{R}^n} (S_j f)^\wedge(\xi) \cdot (S_j \bar{h})^\wedge(\xi) d\xi \\
&= \sum_j \int_{\mathbb{R}^n} |\psi(2^{-j}\xi)|^2 \widehat{f}(\xi) \cdot \widehat{\bar{h}}(\xi) d\xi \\
&= C \int_{\mathbb{R}^n} \widehat{f}(\xi) \cdot \widehat{\bar{h}}(\xi) d\xi \\
&= C \int_{\mathbb{R}^n} f(x)\bar{h}(x) dx
\end{aligned} \tag{5.1.10}$$

for every $f,\ h \in L^2(\mathbb{R}^n)$. When $f \in L^2 \bigcap L^p(\mathbb{R}^n)$ and $h \in L^2 \bigcap L^{p'}(\mathbb{R}^n)$, by (5.1.10) we have

$$\begin{aligned}
C\left|\int_{\mathbb{R}^n} f(x)\bar{h}(x)dx\right| &= \left|\int_{\mathbb{R}^n} \sum_j S_j f(x) \cdot S_j \bar{h}(x) dx\right| \\
&\leq \int_{\mathbb{R}^n} \left(\sum_j |S_j f(x)|^2\right)^{1/2} \left(\sum_j |S_j \bar{h}(x)|^2\right)^{1/2} dx \\
&\leq \left\|\left(\sum_j |S_j f|^2\right)^{1/2}\right\|_p \left\|\left(\sum_j |S_j \bar{h}|^2\right)^{1/2}\right\|_{p'} \\
&\leq C_p' \left\|\left(\sum_j |S_j f|^2\right)^{1/2}\right\|_p \|h\|_{p'},
\end{aligned}$$

where (5.1.7) is used. Thus (5.1.9) holds for $f \in L^p \bigcap L^2(\mathbb{R}^n)$. For general cases, (5.1.9) follows immediately by taking limit. ∎

Remark 5.1.1 *Let $\varphi(x)$ be a nonnegative radial Schwartz function such that $\varphi(x) = 1$ when $|x| \leq \frac{1}{2}$ and $\varphi(x) = 0$ when $|x| \geq 1$. Define*

$$\psi^2(x) = \varphi\left(\frac{\pi}{2}\right) - \varphi(x),$$

then $\sum_j |\psi(2^{-j})|^2 = 1$ for $x \neq 0$. Therefore, we can choose such a ψ to satisfy the condition of Theorem 5.1.4.

Next we give an application of the former result, which is very effective in dealing with operators with rough kernels.

Theorem 5.1.5 *Suppose that $\{\sigma_j\}_{j\in\mathbb{Z}}$ is a sequence of finite Borel measures and satisfies the following conditions:*

(i) there exists $C>0$ such that $\|\sigma_j\|\le C$ for every j;

(ii) there exists $C,\ \alpha>0$ such that

$$|\widehat{\sigma_j}(\xi)|\le C\min\left(\left|2^j\xi\right|^{\alpha},\left|2^j\xi\right|^{-\alpha}\right)$$

for every j;

(iii) there exists $1<p_0<\infty$ such that the maximal operator

$$\sigma^*(f)=\sup_j ||\sigma_j|*f|$$

is bounded on L^{p_0}.

Then operators

$$T(f)=\sum_j \sigma_j*f$$

and

$$S(f)=\left(\sum_j |\sigma_j*f|^2\right)^{1/2}$$

are both bounded on L^p for every p satisfying $\left|\dfrac{1}{p}-\dfrac{1}{2}\right|<\dfrac{1}{2p_0}$.

Proof. Take a nonnegative radial function $\psi\in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\psi)\subset\left\{x\in\mathbb{R}^n:\frac{1}{2}\le|x|\le 2\right\}$$

and

$$\sum_j \psi^2(2^{-j}x)=1\quad(x\ne 0).$$

For $f\in\mathscr{S}(\mathbb{R}^n)$, define S_j by

$$(S_jf)^\wedge(\xi)=\psi\left(2^{-j}\xi\right)\widehat{f}(\xi),\quad j\in\mathbb{Z}.$$

Then

$$Tf(x)=\sum_j\sigma_j*\left(\sum_k S_{j+k}^2f\right)(x):=\sum_k T_kf(x),$$

where

$$T_k f(x) = \sum_j S_{j+k}(\sigma_j * S_{j+k} f)(x).$$

By the Plancherel theorem and the condition (ii) we know (note the condition of the support of ψ)

$$\begin{aligned}
\|T_k f\|_2^2 &= \left\| \sum_j (S_{j+k}(\sigma_j * S_{j+k} f))^\wedge (\cdot) \right\|_2^2 \\
&= \int_{\mathbb{R}^n} \left| \sum_j \widehat{\sigma_j}(\xi) \cdot \psi\left(2^{-j-k}\xi\right)^2 \widehat{f}(\xi) \right|^2 d\xi \qquad (5.1.11) \\
&\leq C \sum_j \int_{|\xi|\sim 2^{j+k}} 2^{-2\alpha|k|} |\widehat{f}(\xi)|^2 d\xi \\
&\leq C 2^{-2\alpha|k|} \|f\|_2^2.
\end{aligned}$$

Let q_0 satisfy $\dfrac{1}{2} - \dfrac{1}{q_0} = \dfrac{1}{2p_0}$, then $p_0 = \left(\dfrac{q_0}{2}\right)'$. Now we will show that, it implies $\{\sigma_j * h_j\} \in L^{q_0}(l^2)$ and

$$\left\| \left(\sum_j |\sigma_j * h_j|^2 \right)^{1/2} \right\|_{q_0} \leq C \left\| \left(\sum_j |h_j|^2 \right)^{1/2} \right\|_{q_0} \qquad (5.1.12)$$

for every $\{h_j\} \in L^{q_0}(l^2)$. In fact, by Hölder's inequality it is easy to deduce that

$$|\sigma_j * h_j(x)|^2 \leq \|\sigma_j\| \cdot \left(|\sigma_j| * |h_j|^2(x) \right). \qquad (5.1.13)$$

Since

$$\left\| \left(\sum_j |\sigma_j * h_j|^2 \right)^{1/2} \right\|_{q_0} = \sup_u \left| \int_{\mathbb{R}^n} \sum_j |\sigma_j * h_j(x)|^2 u(x) dx \right|,$$

where the supremum is taken over all functions $u \in L^{p_0}(\mathbb{R}^n)$ with $\|u\|_{p_0} \le 1$, then by (5.1.13) and Hölder's inequality we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\sum_j |\sigma_j * h_j(x)|^2 u(x)dx\right| &\le C\sum_j \int_{\mathbb{R}^n}\left(|\sigma_j| * |h_j|^2\right)(x)|u(x)|dx\\ &\le C\sum_j\int_{\mathbb{R}^n}|h_j(x)|^2\sigma^*(u)(x)dx\\ &\le C\left\|\left(\sum_j |h_j|^2\right)^{1/2}\right\|_{q_0}^2 \|\sigma^*(u)\|_{p_0}.\end{aligned}$$

Thus (5.1.12) follows from the condition (ii).

For any $g \in L^{p_0'}(\mathbb{R}^n)$, from (5.1.12), Theorem 5.1.4 and Hölder's inequality it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} T_k f(x)g(x)dx\right| &= \left|\sum_j\int_{\mathbb{R}^n}(\sigma_j * S_{j+k}f)(x)(S_{j+k}g)(x)dx\right|\\ &\le \int_{\mathbb{R}^n}\left(\sum_j |\sigma_j * S_{j+k}f|^2\right)^{1/2}\left(\sum_j |S_{j+k}g(x)|^2\right)^{1/2}dx\\ &\le C\left\|\left(\sum_j |\sigma_j * S_{j+k}f|^2\right)^{1/2}\right\|_{p_0}\|g\|_{p_0'}\\ &\le C\left\|\left(\sum_j |S_{j+k}f|^2\right)^{1/2}\right\|_{p_0}\|g\|_{p_0'}\\ &\le C\|f\|_{p_0}\|g\|_{p_0'}.\end{aligned}$$

Thus we have

$$\|T_k f\|_{p_0} \le C\|f\|_{p_0}. \tag{5.1.14}$$

Applying the Riesz-Thörin interpolation theorem to (5.1.11) and (5.1.14), we have that

$$\|T_k f\|_p \le C2^{-\alpha\theta|k|}\|f\|_p,$$

where $0 < \theta < 1$ and p satisfies $\left|\dfrac{1}{p} - \dfrac{1}{2}\right| < \dfrac{1}{2p_0}$. Therefore we obtain that T is of type (p,p).

Finally, we will prove that S is also of type (p, p). Choose a sequence $\varepsilon = \{\varepsilon_j\}$ such that $\varepsilon_j = \pm 1$. Set $T_\varepsilon f(x) = \sum_j \varepsilon_j \sigma_j * f(x)$, then the above discussion implies

$$\|T_\varepsilon f\|_p \leq C_p \|f\|_p \tag{5.1.15}$$

for $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2p_0}$ and C_p independent of ε.

On the other hand, if we set

$$F = \sum_{j=0}^{\infty} a_j r_j \in L^2([0,1]),$$

then $F \in L^p([0,1])$ $(1 < p < \infty)$, where $\{r_j(t)\}$ is the Rademacher function system (see [St3]). Besides, there exist $A_p,\ B_p > 0$ such that

$$A_p \|F\|_p \leq \|F\|_2 = \left(\sum_{j=0}^{\infty} |a_j|^2\right)^{\frac{1}{2}} \leq B_p \|F\|_p.$$

Thus

$$S(f)(x)^p = \left(\sum_j |\sigma_j * f(x)|^2\right)^{p/2} \leq B_p^p \int_0^1 \left|\sum_j r_j(t)\sigma_j * f(x)\right|^p dt. \tag{5.1.16}$$

Integrating both sides of (5.1.16) with respect to x and applying the Fubini theorem as well as (5.1.15), we can derive that S is also of type (p, p). This completes the proof of Theorem 5.1.5. ■

The conclusions of the generalized Littlewood-Paley function g_φ and its discrete form given in this section (see Theorem 5.1.2, Theorem 5.1.3 and Theorem 5.1.4) can be used to discuss the boundedness of Fourier multiplier.

5.2 Weighted Littlewood-Paley theory

In this section, we will discuss weighted Littlewood-Paley theory. That is, we will give the weighted form of Theorem 5.1.3 and Theorem 5.1.4. In order to derive the above conclusions, we will use the weighted normed inequality of vector-valued singular integral operator. More precisely, we will give the weighted form of L^p boundedness (Theorem 2.1.9) of the related vector-valued singular integral operator in Section 2.1, then finish its proof by the dual property. For the definitions and notations of vector-valued function and vector-valued singular integral operator, see Section 2.1. We say that a vector-valued kernel K satisfies (D_r) $(1 \leq r \leq \infty)$ condition, if there exists a sequence $\{c_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} c_k = D_r(K) < \infty$ and

$$\left(\int_{S_k(|y|)} \|K(x-y) - K(x)\|^r_{\mathscr{L}(A,B)} dx\right)^{1/r} \leq c_k |S_k(|y|)|^{-1/r'} \qquad (D_r)$$

for every $k \geq 1$ and $y \in \mathbb{R}^n$, where

$$S_k(|y|) = \left\{x \in \mathbb{R}^n : 2^k|y| < |x| \leq 2^{k+1}|y|\right\}.$$

Remark 5.2.1

(i) If $r = 1$, then the condition (D_1) is just the Hörmander condition (2.1.38).

(ii) If $|x| > 2|y|$ and

$$\|K(x-y) - K(x)\|_{\mathscr{L}(A,B)} \leq C|y|^{\alpha}|x|^{-n-\alpha} \ (\alpha > 0),$$

then K satisfies the condition (D_∞).

(iii) Hölder's inequality implies that if $1 \leq r \leq s \leq \infty$, then $(D_s) \subset (D_r)$.

Next we will give the weighted L^p boundedness of vector-valued singular integral operator.

Theorem 5.2.1 *Suppose that A and B are both reflexive Banach spaces. If there exists $1 < p_0 < \infty$ such that the vector-valued singular integral operator T is bounded from $L^{p_0}(A)$ to $L^{p_0}(B)$, then T can be extended to a bounded operator from $L^p_\omega(A)$ to $L^p_\omega(B)$ when $K(x)$ satisfies the condition $(D_r)(1 < r < \infty)$ and one of the following statements:*

(i) $r' \leq p < \infty$ and $\omega \in A_{p/r'}$;

(ii) $1 < p \leq r$ and $\omega \in A_{p'/r'}$;

(iii) $1 < p < \infty$ and $\omega^{r'} \in A_p$.

Here, for $1 \leq p < \infty$ and nonnegative weight function ω, define $L^p_\omega(A)$ by

$$L^p_\omega(A) = \left\{f : \ \|f\|_{L^p_\omega(A)} = \left(\int_{\mathbb{R}^n} \|f(x)\|^p_A \omega(x) dx\right)^{1/p} < \infty\right\}.$$

First we give the following lemma.

Lemma 5.2.1 *Suppose that K satisfies the condition (D_r) $(1 < r < \infty)$ and T is bounded from $L^{p_0}(A)$ to $L^{p_0}(B)$. Then there exists a constant $C > 0$ such that*

$$M^\sharp(\|Tf\|_B)(x) \leq C \left(M \left(\|f\|_A^{r'} \right) (x) \right)^{1/r'} \tag{5.2.1}$$

for every $f \in L_c^\infty(A)$ and $x \in \mathbb{R}^n$. Here, $L_c^\infty(A)$ represents the set of all A-valued bounded functions with compact supports, M is the Hardy-Littlewood maximal operator and $M^\sharp$ is the Sharp maximal operator.

Proof. Choose $x_0 \in \mathbb{R}^n$. Let U be a ball centered at x_0 and $\widetilde{U} = 2U$. Set

$$b_0 = \int_{y \in (\widetilde{U})^c} K(x_0 - y) f(y) dy \in B.$$

Put $f_1 = f\chi_{\widetilde{U}}$, $f_2 = f - f_1$, then by Theorem 2.1.9 we know that T is also bounded from $L^{r'}(A)$ to $L^{r'}(B)$. Thus

$$\begin{aligned}
&\frac{1}{|U|} \int_U \|Tf(x) - b_U\|_B dx \\
&\leq \frac{1}{|U|} \int_U \|Tf_1(x)\|_B dx + \frac{1}{|U|} \int_U \|Tf_2(y) - b_U\|_B dy \\
&\leq C \left(M \left(\|f\|_A^{r'} \right) (x_0) \right)^{1/r'} \\
&\quad + \sup_{x \in U} \int_{y \in (\widetilde{U})^c} \|K(x - y) - K(x_0 - y)\|_{\mathscr{L}(A,B)} \|f(y)\|_A dy.
\end{aligned}$$

Since K satisfies (D_r) condition, we have

$$\begin{aligned}
&\sup_{x \in U} \int_{y \in (\widetilde{U})^c} \|K(x - y) - K(x_0 - y)\|_{\mathscr{L}(A,B)} \|f(y)\|_A dy \\
&\leq \sum_{k=1}^\infty c_k \left(M \left(\|f\|_A^{r'} \right) (x_0) \right)^{1/r'}.
\end{aligned}$$

By the discussion above we have proved (5.2.1). Hence this completes the proof of Lemma 5.2.1. ∎

Now we turn to the proof of Theorem 5.2.1. First we show this theorem holds when $r' < p < \infty$ and $\omega \in A_{p/r'}$. Using the method in the proof of Theorem 2.2.3, we can get

$$M(\|Tf\|_B) \in L^p(\omega)$$

for $f \in L_c^\infty(A)$. Thus, applying Lemma 2.1.3, Lemma 5.2.1 and Theorem 1.4.3 we have

$$\begin{aligned} \int_{\mathbb{R}^n} \|Tf(x)\|_B^p \omega(x)dx &\leq C \int_{\mathbb{R}^n} \left[M^\sharp(\|Tf\|_B)(x)\right]^p \omega(x)dx \\ &\leq C \int_{\mathbb{R}^n} \left[M\left(\|f\|_A^{r'}\right)(x)\right]^{p/r'} \omega(x)dx \\ &\leq C \int_{\mathbb{R}^n} \|f\|_A^p \omega(x)dx \end{aligned} \tag{5.2.2}$$

for every $f \in L_c^\infty(A)$. If $p = r'$, then $\omega \in A_1$. Hence there exists $p_1 > p$ such that

$$\omega^{\frac{p_1}{p}} \in A_1 \subset A_{p_1/r'}.$$

Consequently, there exists $\varepsilon > 0$ such that $\omega^{p_1(1+\varepsilon)/p} \in A_{p_1/r'}$. Now take $\theta = \dfrac{\varepsilon}{1+\varepsilon}$ and $\widetilde{p_1}$ such that

$$\frac{1}{p} = \frac{\theta}{\widetilde{p_1}} + \frac{1-\theta}{p_1}.$$

It is clear that $1 < \widetilde{p_1} < p < p_1$. Then (5.2.2) yields

$$\|Tf\|_{L_u^{p_1}(B)} \leq C\|f\|_{L_u^{p_1}(A)}, \tag{5.2.3}$$

where $u = \omega^{p_1(1+\varepsilon)/p}$. On the other hand, by Theorem 2.1.9 we know that

$$\|Tf\|_{L^{\widetilde{p_1}}(B)} \leq C\|f\|_{L^{p_1}(A)}.$$

Applying the Stein-Weiss interpolation theorem with change of measure (Lemma 2.2.5) to the above formula and (5.2.3) we see that T is bounded from $L_\omega^p(A)$ to $L_\omega^p(B)$. Therefore we obtain the conclusion under the condition (i). Now we will show that it also holds under the condition (ii) by applying the dual method. Denote by $\widetilde{T}$ the conjugate operator of T. Then $\widetilde{K}$ satisfies $\widetilde{K}(x) = K(-x)$. It is clear that $\widetilde{K}$ satisfies all conditions on K. Since both A and B are reflexive, it follows that $(L^p(A))^* = L^{p'}(A^*)$ and $(L^p(B))^* = L^{p'}(B^*)$. Using the idea in the proof of Theorem 2.2.3, we can finish the proof of Theorem 5.2.1. Finally, the conclusion of Theorem 5.2.1 under the condition (iii) follows from the idea in the proof of Theorem 2.2.4. Thus we have finished the proof of Theorem 5.2.1.

Remark 5.2.2 *The restriction "both A and B are reflexive" can be removed (see [RuRT]) .*

In Section 5.1 we observe that, if $\varphi \in L^1$ satisfies (5.1.2)-(5.1.4), then the generalized Littlewood-Paley function g_φ can characterize $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. It will be shown later that if we strengthen the condition of φ, then g_φ also characterizes weighted L^p spaces $(1 < p < \infty)$.

Theorem 5.2.2 *Suppose that $\varphi \in L^1(\mathbb{R}^n)$ satisfies (5.1.2) and the following conditions*

$$|\varphi(x)| \le \frac{C}{(1+|x|)^{n+1}}, \tag{5.2.4}$$

$$|\nabla\varphi(x)| \le \frac{C}{(1+|x|)^{n+2}}. \tag{5.2.5}$$

For $1 < p < \infty$, if $\omega \in A_p$, then there exist constants C_1 and C_2, independent of f, such that

$$C_1\|f\|_{p,\omega} \le \|g_\varphi(f)\|_{p,\omega} \le C_2\|f\|_{p,\omega} \tag{5.2.6}$$

for every $f \in L^p(\omega)$.

Proof. Let us first prove the second inequality of (5.2.6) by applying Theorem 5.2.1. Take $A = C$, $B = L^2\left(\mathbb{R}_+, \dfrac{dt}{t}\right)$. For $1 < q < \infty$, it follows from Theorem 5.1.2 that g_φ is bounded from $L^q(A)$ to $L^q(B)$. Since $\omega \in A_p$, there exists $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in A_p$. If we set $r' = 1+\varepsilon$, then $1 < r < \infty$. Note that

$$\begin{aligned}
\|\nabla\varphi_t(x)\|_{\mathscr{L}(A,B)} &= \left(\int_0^\infty |\nabla\varphi_t(x)|^2\frac{dt}{t}\right)^{1/2} \\
&\le \left(\int_0^{|x|} \left|\frac{1}{t^{n+1}}\nabla\varphi(\frac{x}{t})\right|^2\frac{dt}{t}\right)^{1/2} \\
&\quad + \left(\int_{|x|}^\infty \left|\frac{1}{t^{n+1}}\nabla\varphi(\frac{x}{t})\right|^2\frac{dt}{t}\right)^{1/2} \\
&\le \frac{C}{|x|^{n+1}}.
\end{aligned} \tag{5.2.7}$$

By (5.2.9), we have that

$$\|\varphi_t(x-y) - \varphi_t(x)\|_{\mathscr{L}(A,B)} \le C\frac{|y|}{|x|^{n+1}}$$

when $|x| > 2|y|$. Thus by Remark 5.2.1 (ii) & (iii) we know that φ_t satisfies the condition (D_r). Then the second inequality of (5.2.6) follows from Theorem 5.2.1.

Now for every $f \in L^2 \bigcap L^p(\omega)$, we have

$$\|f\|_{p,\omega} = \sup \left| \int_{\mathbb{R}^n} f(x)\bar{h}(x)dx \right|, \tag{5.2.8}$$

where the supremum is taken over all functions $h \in L^2 \bigcap L^{p'}(\omega^{-p'/p})$ satisfying $\|h\|_{p',\omega^{-p'/p}} \leq 1$. The property of A_p implies that

$$\omega \in A_p \Longleftrightarrow \omega^{-p'/p} \in A_{p'}.$$

Applying (5.1.6) and the above conclusion, we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f(x)\bar{h}(x)dx \right| &= \left| \int_{\mathbb{R}^n} C \int_0^\infty \varphi_t * f(x) \cdot \varphi_t * \bar{h}(x) \frac{dtdx}{t} \right| \\ &\leq C \int_{\mathbb{R}^n} g_\varphi(f)(x) g_\varphi(\bar{h})(x)dx \\ &\leq C\|g_\varphi(f)\|_{p,\omega} \|g_\varphi(\bar{h})\|_{p',\omega^{-p'/p}} \\ &\leq C\|g_\varphi(f)\|_{p,\omega}. \end{aligned}$$

By (5.2.8) and the density, we get the first inequality of (5.2.6), and then finish the proof of Theorem 5.2.2. ■

The discrete form of Theorem 5.2.2 is the following conclusion, whose proof can be completed by applying Theorem 5.2.1 and the idea in the proof of Theorem 5.1.4. We omit it here.

Theorem 5.2.3 *Suppose that $\psi \in C_0^\infty(\mathbb{R}^n)$ is nonnegative and satisfies*

$$\operatorname{supp}(\psi) \subset \left\{ x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2 \right\},$$

as well as

$$\sum_{j \in \mathbb{Z}} \psi^2 \left(2^{-j}x\right) = 1 \ (x \neq 0).$$

Set $(S_j f)^\wedge(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi)$. Then there exist constants C_1 and C_2 such that

$$C_1\|f\|_{p,\omega} \leq \left\| \left(\sum_{j \in \mathbb{Z}} |\delta_j f(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{p,\omega} \leq C_2\|f\|_{p,\omega} \tag{5.2.9}$$

for $1 < p < \infty$ and $\omega \in A_p$.

Remark 5.2.3

(i) If $\varphi \in \mathscr{S}(\mathbb{R})$ satisfies (5.1.2), then Theorem 5.2.2 holds clearly.

(ii) The method in the proof of Corollary 2.1.5, together with Theorem 5.2.1 and Theorem 5.2.2, yields that the weighted vector-valued inequalities corresponding to (2.1.44) and (2.1.45) still hold for $1 < p < \infty$, $\omega \in A_p$ and

$$\vec{f} = (\cdots, f_{-1},\ f_0,\ f_1, \cdots) \in L^p(\ell^2)(\mathbb{R}^n).$$

Here we omit the details.

5.3 Littlewood-Paley g function with rough kernel

In the first section of this chapter we have seen that, when φ satisfies the vanishing condition (5.1.2), the size condition and the smooth condition (5.1.4), the Littlewood-Paley g function g_φ can characterize the space $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. Especially, g_φ is of type (p, p) $(1 < p < \infty)$ and of weak type $(1, 1)$. Note that g_φ can be regarded as a vector-valued singular integral operator. Thus the discussion about singular integral operators with rough kernels motivates us to put forward the following question: Can we weaken the size condition and the smooth condition on φ but still keep the (p, p) boundedness and the weak $(1, 1)$ boundedness of g_φ? In this section we will consider these problems.

Early in the 1958, while Stein presented and researched Littlewood-Paley g function in higher dimension, he defined the so called Marcinkiewicz integral. Let $\Omega(x)$ be a homogeneous function of degree zero on $\mathbb{R}^n (n \geq 2)$, i.e.,

$$\Omega(\lambda x) = \Omega(x) \tag{5.3.1}$$

for every $\lambda > 0$ and $x \in \mathbb{R}^n$, and it satisfies the vanishing condition

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{5.3.2}$$

If $\Omega(x') \in L^1(\mathbb{S}^{n-1})$, then the Marcinkiewicz integral μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

It is easy to verify that, if we take

$$\varphi(x) = \Omega(x)|x|^{-n+1}\chi_B(x)$$

with $B = \{x \in \mathbb{R}^n : \ |x| \le 1\}$ and χ_B the characteristic function of B, then $\mu_\Omega(f) = g_\varphi(f)$. Therefore μ_Ω is actually a generalized Littlewood-Paley g function.

Stein first proved that, if $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$ $(0 < \alpha \le 1)$, then μ_Ω is of type (p,p) $(1 < p \le 2)$ and of weak type $(1,1)$. By Theorem 2.3.2 and Theorem 2.3.5, we know that when $\Omega \in H^1(\mathbb{S}^{n-1})$ (its definition see Section 2.3) satisfies (5.3.1) and (5.3.2), the singular integral operator T_Ω related to Ω is of type (p,p) $(1 < p < \infty)$. Thus, a natural question is, whether μ_Ω is also of type (p,p) when Ω satisfies the above conditions. The following theorem gives a positive answer.

Theorem 5.3.1 *Suppose that Ω satisfies (5.3.1) and (5.3.2). If $\Omega \in H^1(\mathbb{S}^{n-1})$, then μ_Ω is of type (p,p) $(1 < p < \infty)$.*

Before proving the theorem, we state two lemmas first, their proof can be found in Fan & Pan [FaP2]. For the definition of $H^1(\mathbb{S}^{n-1})$ atom and $H^1(\mathbb{R}^n)$ atom can be found in Section 2.3.

Now we present some notations. Let $a(\cdot)$ be a regular ∞-atom in $H^1(\mathbb{S}^{n-1})$ whose support satisfies

$$\operatorname{supp}(a) \subset \mathbb{S}^{n-1} \bigcap B(\xi', \rho), \qquad (0 < \varphi \le 2, \ \xi' \in \mathbb{S}^{n-1}).$$

When $n \ge 3$, set

$$E_a(s, \xi') = \left(1 - s^2\right)^{\frac{n-3}{2}} \chi_{(-1,1)}(s) \int_{\mathbb{S}^{n-2}} a\left(s, \left(1 - s^2\right)^{\frac{1}{2}} \widetilde{y}\right) d\sigma(\widetilde{y}),$$

and when $n = 2$, set

$$e_a(s, \xi') = \left(1 - s^2\right)^{-\frac{1}{2}} \chi_{(-1,1)}(s) \left[a\left(s, \left(1 - s^2\right)^{\frac{1}{2}}\right) + a\left(s, -\left(1 - s^2\right)^{\frac{1}{2}}\right)\right].$$

Lemma 5.3.1 *There exists a constant $c > 0$, independent of a, such that $cE_a(s, \xi')$ is an ∞-atom on $\mathbb{R}$. That is, cE_α satisfies*

$$\|cE_\alpha\|_\infty \le 1/r(\xi'), \quad \operatorname{supp}(E_\alpha) \subset \left(\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')\right)$$

and

$$\int_{\mathbb{R}} E_a(s, \xi') ds = 0,$$

where $r(\xi') = |\xi|^{-1} |A_\rho \xi|$ *and* $A_\rho(\xi) = (\rho^2 \xi_1, \rho\xi_2, \cdots, \rho\xi_n)$.

Lemma 5.3.2 *For $1 < q < 2$, there exists a constant $c > 0$, independent of a, such that $ce_a(s, \xi')$ is a q-atom on $\mathbb{R}$, the center of whose support is ξ_1' and the radius*

$$r(\xi') = |\xi|^{-1}(\rho^4 \xi_1^2 + \rho^2 \xi_2^2)^{1/2}.$$

Lemma 5.3.3 *Suppose that a is a regular ∞-atom. Set*

$$\sigma_{2^t}(x) = 2^{-t} a\left(\frac{x}{|x|}\right) |x|^{-n+1} \chi_{\{|x| \le 2^t\}}(x)$$

and

$$\sigma^* f(x) = \sup_{t \in \mathbb{R}} ||\sigma_{2^t}| * f(x)|\,.$$

Then there exists a constant $C > 0$, independent of t, a and f, such that

$$\|\sigma_{2^t}\|_1 \le 1$$

and

$$\|\sigma^* f\|_p \le C\|f\|_p$$

for $1 < p \le \infty$.

Proof. Clearly we have $\|\sigma_{2^t}\|_1 \le 1$. On the other hand, it is easy to see $\sigma^* f(x) \le C M_a f(x)$, where M_a is the maximal operator defined by (2.3.1) and C depends only on n. Since $a \in L^1(\mathbb{S}^{n-1})$, $\|\sigma^* f\|_p \le C\|f\|_p$ follows from Theorem 2.3.3, where C is independent of t, a and f. ■

Now we turn to the proof of Theorem 5.3.1. Since $\Omega \in H^1(\mathbb{S}^{n-1})$ satisfies (5.3.2), by Lemma 2.3.1 we can write

$$\Omega(x') = \sum_j \lambda_j a_j(x'),$$

where every a_j is a regular ∞-atom and

$$\sum_j |\lambda_j| \le c\|\Omega\|_{H^1(\mathbb{S}^{n-1})}.$$

Thus we merely need to show that there exists a constant $c > 0$, for each regular ∞-atom a, such that

$$\|\mu_a f\|_p = \left\| \left(\int_0^\infty \left| \int_{|x-y| \le t} \frac{a(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \right\|_p \le C\|f\|_p \quad (5.3.3)$$

for $1 < p \leq \infty$. Without loss of generality, we may assume

$$\mathrm{supp}(a) \subset \mathbb{S}^{n-1} \bigcap B(\ell, \rho),$$

where $\ell = (1, 0, \cdots, 0)$. Consider the case $n \geq 3$ first. Chooose a nonnegative and radial C_0^∞ function ψ such that

$$0 \leq \psi \leq 1, \quad \mathrm{supp}(\psi) \subset \left\{ x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2 \right\}$$

and

$$\int_0^\infty \frac{\psi(t)}{t} dt = 1.$$

We define Φ and Δ by

$$\widehat{\Phi}(\xi) = \psi(A_\rho \xi)$$

and

$$\widehat{\Delta}(\xi) = \psi(\xi),$$

respectively. (Here $A_\rho \xi = (\rho^2 \xi_1, \rho\xi_2, \cdots, \rho\xi_n)$, see Lemma 5.3.1). Write

$$\Phi_t(x) = \frac{1}{t^n} \Phi\left(\frac{x}{t}\right), \qquad \Delta_t(x) = \frac{1}{t^n} \Delta\left(\frac{x}{t}\right).$$

Then it is easy to see that

$$\widehat{\Phi_t}(\xi) = \psi(t A_\rho \xi), \qquad \widehat{\Delta_t}(\xi) = \psi(t\xi)$$

and

$$\Phi_t(x) = \frac{1}{\rho^{n+1}} t^{-n} \Delta\left(\frac{A_{\frac{1}{\rho}} x}{t}\right).$$

Clearly for any $f \in \mathscr{S}(\mathbb{R}^n)$, we have that

$$f(x) \sim \int_0^\infty \frac{\Phi_t * f(x)}{t} dt \sim \int_{-\infty}^\infty \Phi_{2^t} * f(x) dt. \tag{5.3.4}$$

Set

$$g_\Phi(f)(x) = \left(\int_{-\infty}^\infty |\Phi_{2^t} * f(x)|^2 dt \right)^{1/2} \sim \left(\int_0^\infty |\Phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Then there exists a constant $C > 0$, independent of ρ and f, such that

$$\|g_\Phi(f)\|_p \leq C \|f\|_p. \tag{5.3.5}$$

In fact, we see that

$$\Phi_t * f(x) \sim \int_{\mathbb{R}^n} t^{-n}\Delta\left(\frac{y}{t}\right) f\left(A_\rho\left(A_{\frac{1}{\rho}}x - y\right)\right) dy = \Delta_t * h\left(A_{\frac{1}{\rho}}x\right),$$

where $h(x) = f(A_\rho x)$. Note that

$$\int_{\mathbb{R}^n} \Delta(x)dx = \widehat{\Delta}(0) = \psi(0) = 0.$$

Applying Theorem 5.1.3 implies that

$$\begin{aligned}
\|g_\Phi(f)\|_p &\le c\left(\int_{\mathbb{R}^n}\left(\int_0^\infty \left|\Delta_t * h\left(A_{\frac{1}{\rho}}x\right)\right|^2 \frac{dt}{t}\right)^{p/2} dx\right)^{1/p} \\
&= c\rho^{\frac{n+1}{p}}\left\|\left(\int_0^\infty \left|\Delta_t * h\left(A_{\frac{1}{\rho}}x\right)\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_p \\
&\le c\rho^{\frac{n+1}{p}}\|h\|_p = c\|f\|_p.
\end{aligned}$$

Notice that

$$\begin{aligned}
\mu_a f(x) &= \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{a(x-y)}{|x-y|^n} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&\le c\left(\int_{-\infty}^\infty |\sigma_{2^t} * f(x)|^2 dt\right)^{1/2}.
\end{aligned}$$

By (5.3.4) and Minkowski's inequality, we conclude that

$$\begin{aligned}
\mu_a f(x) &\le c\int_{-\infty}^\infty \left(\int_{-\infty}^\infty |\Phi_{2^{s+t}} * \sigma_{2^t} * f(x)|^2 dt\right)^{1/2} ds \\
&:= c\int_{-\infty}^\infty H_s f(x) ds.
\end{aligned}$$

By Minkowski's inequality again, we have

$$\|\mu_a f\|_p \le c\int_{-\infty}^\infty \|H_s f\|_p ds. \tag{5.3.6}$$

Under the condition of the theorem, if we can illustrate that there exist constants $c > 0$ (independent of s, f and a), $\theta,\ \eta > 0$, such that

$$\|H_s f\|_p \le \begin{cases} c2^{-\theta s}\|f\|_p & \text{when } s > 0, \\ c2^{\eta s}\|f\|_p & \text{when } s < 0, \end{cases} \tag{5.3.7}$$

then (5.3.3) follows from (5.3.6) and (5.3.7). Therefore, the proof of Theorem 5.3.1 can be reduced to the proof of (5.3.7), which is a direct consequence of two estimates below and the Riesz-Thörin interpolation theorem of sublinear operators (see Calderón-Zygmund [CaZ2]):

$$\|H_s f\|_p \le c\|f\|_p \tag{5.3.8}$$

$$\|H_s f\|_2 \le \begin{cases} c \cdot 2^{-2s}\|f\|_2 & \text{when } s > 0, \\ c \cdot 2^{s/4}\|f\|_2 & \text{when } s < 0. \end{cases} \tag{5.3.9}$$

In the above two formulas, $c > 0$ is independent on s, a and f.

Let us first prove (5.3.8). For $1 < p < 2$, since a is an ∞-atom, it follows that $\|\sigma_{2^t}\|_1 \le 1$ from Lemma 5.3.3. Set

$$G_{s+t}(x) = \Phi_{2^{s+t}} * f(x).$$

Then we have

$$\left\| \int_{-\infty}^{\infty} \sigma_{2^t} * G_{s+t}(\cdot)dt \right\|_1 \le \left\| \int_{-\infty}^{\infty} |G_t(\cdot)|dt \right\|_1. \tag{5.3.10}$$

On the other hand, applying Lemma 5.3.3 yields that

$$\left\| \sup_{t\in\mathbb{R}} |\sigma_{2^t} * G_{s+t}| \right\|_q \le \left\| \sigma^* \left(\sup_{t\in\mathbb{R}} |G_t| \right) \right\|_q \le c \left\| \sup_{t\in\mathbb{R}} |G_t| \right\|_q \tag{5.3.11}$$

for $1 < q < \infty$. If we set

$$TG_{s+t}(x) = \sigma_{2^t} * G_{s+t}(x),$$

then (5.3.10) and (5.3.11) show that T is bounded on $L^1\left(L^1(\mathbb{R}), \mathbb{R}^n\right)$ and $L^q\left(L^\infty(\mathbb{R}), \mathbb{R}^n\right)$, respectively. For $1 < p < 2$, take

$$\frac{1}{q} = \frac{2}{p} - 1.$$

Then by applying the interpolation theorem mentioned above, we have that T is bounded on $L^p(L^2(\mathbb{R}), \mathbb{R}^n)$. That is,

$$\left\| \left(\int_{-\infty}^{\infty} |\sigma_{2^t} * G_{s+t}(\cdot)|^2 dt \right)^{1/2} \right\|_p \le C \left\| \left(\int_{-\infty}^{\infty} |G_t(\cdot)|^2 dt \right)^{1/2} \right\|_p,$$

where C is independent of s, a and f. This inequality together with (5.3.5) implies that (5.3.8) holds for $1 < p < 2$. If $p > 2$, set $q = \left(\frac{p}{2}\right)'$, then there exists $\omega \in L^q$ with $\|\omega\|_q \leq 1$, such that

$$\|H_s f\|_p^2 = \left| \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |\Phi_{2^{s+t}} * \sigma_{2^t} * f(x)|^2 \omega(x) dt dx \right|.$$

By Hölder's inequality, Lemma 5.3.3 and (5.3.5), we have that

$$\|H_s f\|_p^2 \leq \|g_\Phi(f)\|_p^2 \|\sigma^*(|\omega|)\|_q \leq C\|f\|_p^2.$$

This formula implies that (5.3.8) holds for $p > 2$, and C is independent of s, a and f.

Now we estimate (5.3.9). By the Plancherel theorem, we have that

$$\|H_s f\|_2^2 \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \psi(2^{s+t} A_\rho \xi)^2 2^{-2t} |J_t(\xi)|^2 d\xi dt,$$

where

$$J_t(\xi) = \int_{|y| \leq 2^t} \frac{a(y')}{|y|^{n-1}} e^{-2\pi i \xi \cdot y} dy.$$

For $\xi \neq 0$, choose a rotation O such that $O(\xi) = |\xi| \cdot \ell = |\xi|(1, 0, \cdots, 0)$. Set $y' = (u, y_2', \cdots, y_n')$, then

$$J_t(\xi) = \int_0^{2^t} \int_{\mathbb{S}^{n-1}} a(O^{-1}y') e^{-2\pi i r|\xi|u} d\sigma(y') dr,$$

where O^{-1} denotes the reverse rotation of O. Clearly, $a(O^{-1}y')$ is a regular ∞-atom whose support lies in $\mathbb{S}^{n-1} \bigcap B(\xi', \rho)$. Thus

$$J_t(\xi) = \int_0^{2^t} \int_{\mathbb{R}} E_a(u, \xi') e^{-2\pi i r|\xi|u} du dr.$$

By Lemma 5.3.1, we have that

$$|J_t(\xi)| \leq c 2^{2t} |\xi| \int_{\mathbb{R}} |E_a(u, \xi')| |u - \xi_1'| du \leq c 2^{2t} |A_\rho \xi|. \tag{5.3.12}$$

On the other hand, we obtain that

$$\begin{aligned} |J_t(\xi)| &\leq C \int_2^{2^t|\xi|} \left| \int_{\mathbb{R}} E_a(u, \xi') e^{-2\pi i r u} du \right| \frac{dr}{|\xi|} \\ &\leq C \int_2^{2^t|\xi|} \left| \widehat{E_a}(r) \right| \frac{dr}{|\xi|}. \end{aligned}$$

Applying Hölder's inequality, Hausdorff-Young's inequality and Lemma 5.3.1, we know that

$$|J_t(\xi)| \le C|\xi|^{-1}\left(2^t|\xi|\right)^{3/4}\|E_n\|_{L^{\frac{4}{3}}(\mathbb{R})} \le c2^{3t/4}|A_\rho\xi|^{-1/4}.$$

This inequality together with (5.3.12) implies

$$|J_t(\xi)| \le C\min\left\{2^{3t/4}|A_\rho\xi|^{-1/4},\ 2^{2t}|A_\rho\xi|\right\}. \tag{5.3.13}$$

It is clear that the constant C in (5.3.13) is independent of s, a and f. By (5.3.13), when $s>0$, we have

$$\begin{aligned}\|H_sf\|_2^2 &\le C\int_{-\infty}^{\infty}\int_{2^{-s-t-1}\le|A_\rho\xi|\le 2^{-s-t+1}}\left|\widehat{f}(\xi)\right|^2 2^{2t}|A_\rho\xi|^2d\xi dt\\ &\le C2^{-2s}\|f\|_2^2.\end{aligned}$$

When $s<0$, by applying (5.3.13) we obtain that

$$\|H_sf\|_2^2 \le c2^{s/2}\|f\|_2^2.$$

Thus we get (5.3.9), and finish the proof of Theorem 5.3.1 when $n\ge 3$. For $n=2$, we just need to replace Lemma 5.3.1 by Lemma 5.3.2.

Remark 5.3.1

(i) Since $L(\log^+ L)(\mathbb{S}^{n-1}) \subsetneq H^1(\mathbb{S}^{n-1})$, *the conclusion of Theorem 5.3.1 still holds for* $\Omega\in L(\log^+ L)(\mathbb{S}^{n-1})$.

(ii) When $\Omega\in H^1(\mathbb{S}^{n-1})$ *satisfies (5.3.1) and (5.3.2), whether* μ_Ω *is of weak type* $(1,1)$ *is still a question.*

The following theorem will give another result on L^p boundedness of Marcinkiewicz integral with rough kernel.

Theorem 5.3.2 *Suppose that* Ω *satisfies (5.3.1) and (5.3.2). If*

$$\Omega\in L(\log^+ L)^{1/2}(\mathbb{S}^{n-1}),$$

then μ_Ω *is of type* (p,p) $(1<p<\infty)$.

First we state a lemma, which essentially is the continuous form of Theorem 5.1.5. Its proof will be put off after the proof of Theorem 5.3.2. Suppose $\tau=\{\tau_t : t\in\mathbb{R}\}$ is a family of measures on $\mathbb{R}^n$. Define operators Δ_τ and τ^* by

$$\Delta_\tau(f)(x) = \left(\int_{\mathbb{R}}|\tau_t * f(x)|^2dt\right)^{1/2}, \tag{5.3.14}$$

$$\tau^*(f)(x) = \sup_{t\in\mathbb{R}}\left(|\tau_t| * |f|\right)(x). \tag{5.3.15}$$

Lemma 5.3.4 *Suppose that $a \geq 2,\ A > 0,\ \gamma > 0,\ q > 1$ and $C_q > 0$. Let a family of measures $\{\tau_t : t \in \mathbb{R}\}$ on $\mathbb{R}^n$ satisfy*

(i) $\|\tau_t\| \leq A, \quad t \in \mathbb{R}$;

(ii) $|\widehat{\tau}_t(\xi)| \leq A\left(\min\left\{a^t|\xi|,\ \left(a^t|\xi|\right)^{-1}\right\}\right)^{\gamma/\log a}$, for every $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

(iii) $\|\tau^*(f)\|_q \leq C_q A\|f\|_q$, for every $f \in L^q(\mathbb{R}^n)$.

Then for p satisfying

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2q},$$

there exists a constant C_p, independent of f and a, such that

$$\|\Delta_\tau(f)\|_p \leq C_p\|f\|_p$$

for $f \in L^p(\mathbb{R}^n)$.

The proof of Theorem 5.3.2. For $k \in \mathbb{N}$, Set

$$E_k = \left\{y' \in \mathbb{S}^{n-1} : 2^{k-1} \leq |\Omega(y')| < 2^k\right\}$$

and

$$\Omega_k(y') = \Omega(y')\chi_{E_k}(y') - \int_{E_k} \Omega(x')d\sigma(x').$$

Then

$$\int_{\mathbb{S}^{n-1}} \Omega_k(x')d\sigma(x') = 0 \tag{5.3.16}$$

for all $k \in \mathbb{N}$. Let

$$\bigwedge = \left\{x \in \mathbb{N} : |E_k| > 2^{-4k}\right\},$$

where $|E_k|$ represents the measure of E_k on $\mathbb{S}^{n-1}$ induced by Lebesgue measure. Set

$$\Omega_0 = \Omega - \sum_{k \in \bigwedge} \Omega_k,$$

then it is easy to verify that $\Omega_0 \in L^2(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}} \Omega_0(x')d\sigma(x') = 0.$$

For $k \in \bigwedge$, define a family of measures $\tau^{(k)} = \{\tau_{k,t} : t \in \mathbb{R}\}$ on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f d\tau_{k,t} = 2^{-kt} \int_{|y| \leq 2^{kt}} \frac{\Omega_k(y)}{|y|^{n-1}} f(y)dy.$$

Now set

$$a_k = 2^k, \quad A_k = 2\int_{E_k} |\Omega(y')|d\sigma(y'), \quad \gamma = \frac{\log 2}{6}.$$

Then for $t \in \mathbb{R},\ \xi \in \mathbb{R}^n$ and $p > 1$, we have the following results:

$$\begin{aligned}
&(i)\ \|\tau_{k,t}\| \le A_k,\\
&(ii)\ |\widehat{\tau_{k,t}}(\xi)| \le A_k \left(a_k^t|\xi|\right)^{\gamma/\log a_k},\\
&(iii)\ |\widehat{\tau_{k,t}}(\xi)| \le CA_k \left(a_k^t|\xi|\right)^{\gamma/\log a_k},\\
&(iv)\ \left\|\left(\tau^{(k)}\right)^*\right\|_{L^p\to L^p} \le C_pA_k.
\end{aligned} \tag{5.3.17}$$

In fact, (5.3.17) (i) is obvious. It follows from (5.3.16) that

$$|\widehat{\tau_{k,t}}(\xi)| \le A_k \left(a_k^t|\xi|\right).$$

This formula together with $|\widehat{\tau_{k,t}}(\xi)| \le \|\tau_{k,t}\| \le A_k$ implies (5.3.17) (ii) (Note that $\gamma/\log a_k < 1$). Since

$$\left(\tau^{(k)}\right)^*(f)(x) \le \int_{\mathbb{S}^{n-1}} \left|\Omega_k\left(y'\right)\right| M_{y'}f(x)d\sigma(y'),$$

where $M_{y'}$ is the Hardy-Littlewood maximal operator along y', we have that (5.3.17) (iv) holds. Next we will show (5.3.17) (iii), and the method is analogous to that in the proof of (2.2.4). By Hölder's inequality, we obtain that

$$\begin{aligned}
|\widehat{\tau_{k,t}}(\xi)|^2 &\le 2^{-kt}\int_0^{2^{kt}} \left|\int_{\mathbb{S}^{n-1}} \Omega_k(y')e^{-2\pi ir\langle y',\xi\rangle}d\sigma(y')\right|^2 dr\\
&\le C\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \Omega_k(y')\overline{\Omega_k(x')}\\
&\quad\times\left(\sum_{j=-\infty}^{0} 2^j\int_{2^{kt+j-1}}^{2^{kt+j}} e^{-2\pi ir\langle y'-x',\xi\rangle}\frac{dr}{r}\right)d\sigma(y')d\sigma(x').
\end{aligned}$$

Set

$$I = \int_{2^{kt+j-1}}^{2^{kt+j}} e^{-2\pi ir\langle y'-x',\xi\rangle}\frac{dr}{r},$$

then clearly we have $|I| \le \log 2$. On the other hand, we see that

$$|I| \le C\left(2^{kt+j-1}|\xi|\right)^{-1}\left|\langle y'-x',\xi'\rangle\right|^{-1},$$

where C is independent of k, i, j, x', y' and ξ. Thus

$$|I| \leq C \left(2^{kt+j-1}|\xi|\right)^{-\alpha} \left|\langle y' - x', \xi' \rangle\right|^{-\alpha} \tag{5.3.18}$$

for $0 < \alpha < 1$. Note that

$$\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \frac{1}{|\langle y' - x', \xi' \rangle|^{\alpha}} d\sigma(y')d\sigma(x') \leq C_1.$$

Thus from (5.3.18) it follows that (5.3.17) (iii) holds.

By Minkowski's inequality, we have

$$\mu_\Omega(f) \leq \mu_{\Omega_0}(f) + \sum_{k\in\bigwedge} (k\log 2)^{1/2} \triangle_{\tau^{(k)}}(f).$$

Since Theorem 5.3.1 implies that μ_{Ω_0} is of type (p,p) $(1 < p < \infty)$, by (5.3.17) and Lemma 5.3.4 we obtain that

$$\begin{aligned}\|\mu_\Omega(f)\|_p &\leq C_p \left(1 + \sum_{k\in\bigwedge} \sqrt{k} A_k\right) \|f\|_p \\ &\leq C_p \left(1 + \|\Omega\|_{L(\log^+ L)^{1/2}}\right) \|f\|_p.\end{aligned}$$

Thus we finish the proof of Theorem 5.3.2.

Next we give the proof of Lemma 5.3.4. The idea is analogous to that of (5.3.6). Here we merely give the main steps.

First take a nonnegative radial function ψ in $\mathscr{S}(\mathbb{R}^n)$, which satisfies

$$0 \leq \psi \leq 1, \quad \operatorname{supp}(\psi) \subset \left\{x \in \mathbb{R}^n : \frac{4}{5a} \leq |x| \leq \frac{5a}{4}\right\}$$

and

$$\int_0^\infty \psi(t)\frac{dt}{t} = 2\log a.$$

Set $\widehat{\Psi}(\xi) = \psi(|\xi|^2)$. Then for $t > 0$, $\widehat{\Psi_t}(\xi) = \psi(|t\xi|^2)$ and

$$f(x) = \int_{-\infty}^\infty \Psi_{a^t} * f(x)dt$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. By Minkowski's inequality, we have

$$\Delta_\tau(f)(x) \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\Psi_{a^{s+t}} * \tau_t * f(x)|^2 dt\right)^{1/2} ds.$$

If we set

$$H_s(f)(x) = \left(\int_{\mathbb{R}} |\Psi_{a^{s+t}} * \tau_t * f(x)|^2 dt\right)^{1/2},$$

then it suffices to prove that there exist constants $C > 0$, γ and $\theta > 0$, independent of s and f, such that

$$\|H_s f\|_p \leq \begin{cases} CAe^{-s\gamma}\|f\|_p & \text{if } s > 0 \\ CAe^{s\theta}\|f\|_p & \text{if } s < 0. \end{cases}$$

In order to prove this, we merely need to show

$$\|H_s f\|_p \leq CA\|f\|_p \tag{5.3.19}$$

and

$$\|H_s f\|_2 \leq \begin{cases} CAe^{-s\gamma}\|f\|_2 & \text{if } s > 0 \\ CAe^{s\theta}\|f\|_2 & \text{if } s < 0. \end{cases} \tag{5.3.20}$$

Applying the condition (i) and (iii) of Lemma 5.3.4, as well as the idea in the proof of (5.3.8), we can obtain (5.3.19). The proof of (5.3.20) can be completed by using the Plancherel theorem and the condition (ii).

Remark 5.3.2

$H^1(\mathbb{S}^{n-1})$ and $L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$ both contain $L\log^+ L(\mathbb{S}^{n-1})$ as a subspace. However, there are examples showing that, neither $H^1(\mathbb{S}^{n-1})$ nor $L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$ contains the other.

Finally, we state a result on the weighted L^p boundedness of Marcinkiewicz integral with rough kernel.

Theorem 5.3.3 *Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ $(q > 1)$ satisfies (5.3.1) and (5.3.2). If p, q, ω satisfy one of the following conditions*

(a) $q' < p < \infty$ and $\omega \in A_{p/q'}$;

(b) $1 < p < q$ and $\omega^{1-p'} \in A_{p'/q'}$;

(c) $1 < p < \infty$ and $\omega^{q'} \in A_p$,

then there exists a constant C, independent of f, such that $\|\mu_\Omega f\|_{p,\omega} \leq C\|f\|_{p,\omega}$.

5.4 Notes and references

The Littlewood-Paley g function in one dimension was first introduced by Littlewood and Paley [LiP] in studying the dyadic decomposition of Fourier series. And this theory in high dimension was established by Stein [St1]. Theorem 5.1.1 is due to Stein.

The conclusion of Theorem 5.2.1 under the condition (i) or (ii) comes from Rubio de Francia, Ruiz and Torrea [RuRT]. The assumption "both A and B are reflexive spaces" in the theorem can be removed. The weighted Littlewood-Paley theory was first studied by Kurtz [Ku]. The proof of Theorem 5.2.3 can be found in Watson and Wheeden [WatW].

Theorem 5.3.1 and its proof are taken from Ding, Fan and Pan [DiFP2]. In 2003, Xu, Chen and Ying [XuCY] also gave the proof of Theorem 5.3.1 using another method. For the weak type (1,1) bounds of Marcinkiewicz integrals with rough kernels μ_Ω, Fan and Sato [FaS] proved that if $\Omega \in L(log^+ L)(\mathbb{S}^{n-1})$ satisfies (5.3.1) and (5.3.2), then μ_Ω is of weak type (1,1). The conclusion of Theorem 5.3.2 for $p = 2$ was first proved by Walsh [Wal], and the proof for $1 < p < \infty$ was given by Al-Salman, Al-Qassem, Cheng and Pan [AACP]. And Walsh [Wal] showed that if $\Omega \in L(log^+ L)^{1/2-\varepsilon}(\mathbb{S}^{n-1})$ for $0 < \varepsilon < 1/2$ and satisfies (5.3.2), then there exists an $f \in L^2(\mathbb{R}^n)$ such that $\mu_\Omega f \notin L^2(\mathbb{R}^n)$. The weighted boundedness of Marcinkiewicz integral was first studied by Torchinsky and Wang [ToW]. They proved that if $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})(0 < \alpha \le 1)$ and $\omega \in A_p(1 < p < \infty)$, then μ_Ω is bounded on $L^p(\omega)$. Theorem 5.3.3 was obtained by Ding, Fan and Pan [DiFP1], and independently by Duoandikoetxea and Seijo [DuS]. In this chapter, we do not mention the boundedness of the commutator generated by Marcinkiewicz integral μ_Ω and a BMO function b. Here we mere mention two results. Torchinsky and Wang [ToW] proved L^p-boundedness of $[\mu_\Omega, b]$ with $1 < p < \infty$ if $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})(0 < \alpha \le 1)$. And Ding, Lu and Yabuta [DiLY] proved L^p-boundedness of the commutator if $\Omega \in L^q(\mathbb{S}^{n-1})(1 < q \le \infty)$.

As space is limited, we do not mention the boundedness of Marcinkiewicz integral with homogeneous kernels on Hardy spaces and Companato spaces. For related results, we refer to Ding, Lu and Xue [DiLX1], [DiLX2].

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